first choice while A receives the most votes for second choice. Then

$$B_3 = A_2 \qquad \text{and} \qquad B_2 = A_3.$$

It is now impossible for B to win. For if B is to win we must have

$$xA_1 + yA_2 + zA_3 < xA_1 + yA_3 + zA_2,$$

which reduces to

$$y < z$$

and this is contrary to hypothesis. In like manner we can show that *if A and B have an equal number of votes for second or third choice and A has the greater number of votes for first choice, then it is impossible for B to win.*

There remains the case where any two candidates A and B receive an unequal number of votes for first, second, and third choice. We will now suppose that A receives the most votes for first choice. Under this hypothesis it is clear that the x-axis lies within the region $\bar{a}$, for the weights 1, 0, 0 are such as to make A win.

Designate by *ab, ac, bc* the lines of intersection of the plane

$$x = \text{any real positive constant} \tag{2}$$

with the planes AB, AC, BC. Designate by O' and P its intersections with the x-axis and with the line $x = y = z$. (See figures 2 and 3.) Let $O'z'$ and $O'y'$

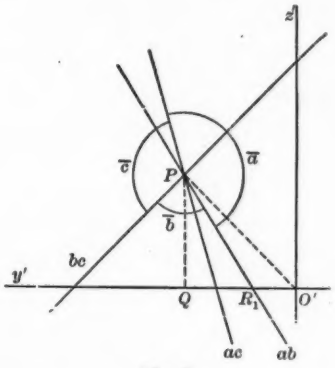


Fig. 2.

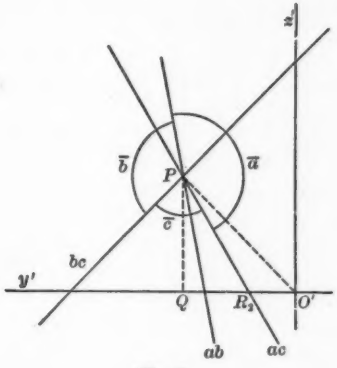


Fig. 3.

be the intersections of the plane (2) with the planes $y = 0$ and $z = 0$. Draw QP perpendicular to $O'y'$. In order that $x > y > z$, we must choose as weights the coördinates of some point within the triangle $O'PQ$. Since the plane AB contains the line $x = y = z$, the line ab must contain the point P. The region $\bar{b}$ and the x-axis are on opposite sides of the plane AB. Hence in order that B may win over A it is necessary that the line ab have a positive slope greater than unity but not infinite. In like manner in order that C may win over A

it is necessary that the line *ac* have a positive slope greater than unity but not infinite.

Suppose the slopes of *ab* and *ac* both positive and greater than unity. If the slope of *ab* [or *ac*] is less than that of *ac* [or *ab*] then *B* [or *C*] may win over both *A* and *C* [or *B*] by choosing as weights the coördinates of any point within the region PQR_1 [or PQR_2]. From Fig. 2 (or Fig. 3) in order that *C* [or *B*] may win over *B* [or *C*] and *A* it is necessary that the line *bc* have a positive slope greater than unity but not infinite and also greater than the slope of *ac* [or *ab*].

Thus, in order that either one of B or C at will may win over the other two candidates, A having the greatest number of votes for first choice, it is necessary that each of the three lines ab, ac, bc have a positive slope greater than unity but not infinite and that of these three lines the slope of bc be the greatest.

These conditions are also sufficient. *Illustration:*

	Choice			
	1st	2d	3d	
A	9	2	9	
B	4	11	5	
C	7	7	6	Winner
Weights	4	2	1	*A*
	6	5	1	*B*
	3	2	1	*C*

A DIRECT PROOF OF DE MOIVRE'S FORMULA.

By S. LEFSCHETZ, University of Kansas.

The proposition which will be proved here, and which is practically equivalent to De Moivre's noted theorem, may be stated thus:

If X, Y, Z are three complex numbers of modulus unity, and such that their arguments x, y, z have a zero sum, then $XYZ = 1$.

Let $X = a + ia'$, $Y = b + ib'$, $Z = c + ic'$, and denote their conjugates by $\overline{X}$, $\overline{Y}$, $\overline{Z}$. The moduli being unity, we have $\overline{X} = 1/X$, $\overline{Y} = 1/Y$, $\overline{Z} = 1/Z$. Let *A* and *B* be the representative points of *X* and $\overline{Y}$, using Argand's diagram, *O* the origin, *E* the foot of the perpendicular *AE* from *A* to *OB*. As angle $BOA = (x + y)$, it follows that $OE = -c$, $EA = -c'$. The coefficients of the equations of the lines *OB* and *AE* are rational in the coördinates of *A*, *B*, that is, in *a*, *b*, *a'*, *b'*, and therefore the same holds for the coördinates of their intersection *E*. When these are known, the ratio $OE/OB = -c/1$ is determined rationally. Hence *c*, and similarly *c'*, can be determined rationally with respect to *a*, *b*, *a'*, *b'*. Therefore.

(1) $$Z = c + ic' = f(a, b, a', b'),$$

where f is a rational function. But

$$a = \frac{1}{2}(X + \bar{X}) = \frac{1}{2}\left(X + \frac{1}{X}\right), \qquad a' = \frac{1}{2i}\left(X - \frac{1}{X}\right), \qquad b = \frac{1}{2}\left(Y + \frac{1}{Y}\right),$$

$$b' = \frac{1}{2i}\left(Y - \frac{1}{Y}\right).$$

Substituting in (1), it can be replaced by a relation such as

$$F(X, Y, Z) = 0, \tag{2}$$

where F is a polynomial in X, Y, Z, of the first degree in Z. When the three variables are arbitrarily permuted, five other relations of the same type are obtained which, for reasons of symmetry, must be equally satisfied. But between X, Y, Z, only one relation can exist, since two of them are arbitrary. Hence the polynomials at the left of the relations analogous to (2), which have just been considered, have a common divisor $\phi(X, Y, Z)$ symmetric with respect to X, Y, Z, and we have $\phi = 0$. As ϕ, like F itself, is necessarily of the first degree in Z, it is linear in the three variables and we have a relation such as

$$A \cdot XYZ + B(XY + YZ + ZX) + C(X + Y + Z) + D = 0, \tag{3}$$

A, B, C being coefficients still to be determined.

If the arguments are changed in sign, X, Y, Z are changed into their conjugates, and (3) is still satisfied. Hence A, B, C, D are real. As the conjugates are also the reciprocals in this case, coefficients in (3) equidistant from the extremes can differ only in sign.

Hence $D = kA$, $C = kB$, $k = \pm 1$. But for $x = y = \pi/2$, we have $z = -\pi$, $X = Y = i$, $Z = -1$. Therefore,

$$A(1 + k) + B[-1 - 2i + k(2i - 1)] = 0.$$

Equating to zero real and complex parts respectively, we obtain

$$(1 + k)(A - B) = 0, \qquad (1 - k)B = 0.$$

Hence, either: (a) $k = +1$, $A = B$; or (b) $k = -1$, $B = 0$, when we can take $A = 1$. But if $x = y = z = 0$, then $X = Y = Z = 1$. Substituting in (3), we have

$$(A + 3B)(1 + k) = 0,$$

which is incompatible with assumption (a).

Hence $k = -1$, $B = 0$, and therefore $XYZ = 1$, as was to be proved.

Since

$$X = \cos x + i \sin x, \qquad Y = \cos y + i \sin y, \qquad z = -(x + y),$$

we have

$$(\cos x + i \sin x)(\cos y + i \sin y) = 1/Z = \overline{Z} = \cos(-x - y) - i \sin(-x - y)$$
$$= \cos(x + y) + i \sin(x + y),$$

which is De Moivre's formula. From this the trigonometric addition or subtraction formulas are easily derived in all their generality.

SAILING TO WINDWARD.

BY W. E. BYERLY, Harvard University.

The problem of sailing to a windward goal is usually the simple problem of *getting to windward*. For, if we find the straight path of least time from the boat to a perpendicular to the wind's direction, by tacking at the proper point we shall reach any goal in that perpendicular in the time it would take to reach the perpendicular had we continued on the original course.

The velocity of the boat on any course is some function of θ depending on the model and sailing qualities of the boat, where θ is the angle which the course makes with the direction of the wind, and we shall suppose this function given, and shall represent it by $F(\theta)$.

The time required to get the distance a to windward is the distance sailed, $a \sec \theta$, divided by the velocity, $F(\theta)$, and we wish to make this time a minimum.

Let $u = a \sec \theta / F(\theta)$. Then

$$\frac{du}{d\theta} = \frac{F(\theta)\, a \sec\theta \tan\theta - a \sec\theta F'(\theta)}{[F(\theta)]^2} = 0, \qquad \text{where} \qquad F'(\theta) = \frac{d}{d\theta} F(\theta),$$

or

$$F(\theta) \tan\theta - F'(\theta) = 0. \tag{1}$$

The solution of this equation will give us the angle which our course on each tack should make with the direction of the wind, whether we are using sails alone or sails and auxiliary motor.

In any concrete case equation (1) can be solved by "trial and error" with the aid of a three-place trigonometric table (preferably giving the angles in radians as well as in degrees) with sufficient accuracy for all practical purposes.

The function $F(\theta)$ ought to be determined by experiment for every boat, but as a first approximation, in the case of a sail boat, it may be taken as equal to $k(\theta - \alpha)$, where α, "the angle of repose," is the angle with the direction of the wind within which the boat will not sail; an angle, by the way, that is very easily discovered by experimenting with the boat.

Equation (1) becomes $k(\theta - \alpha) \tan \theta - k = 0$ or

$$\theta - \alpha = \operatorname{ctn} \theta. \tag{2}$$

The θ found from this equation is the angular distance off the wind which we should sail in going to windward, and is evidently independent of the wind's velocity.

For a rather sluggish boat, for which α is 30°, θ is about 60°.

Let us now suppose that the sail boat has a gasolene engine which can give her a velocity v. Then with sail and engine, $F(\theta) = v + k(\theta - \alpha)$, and equation (1) becomes $[v + k(\theta - \alpha)] \tan \theta - k = 0$ or

$$\frac{v}{k} + (\theta - \alpha) = \operatorname{ctn} \theta. \tag{3}$$

Here θ is no longer the same for all breezes but k is easily determined by shutting off the motor and noting the speed on any convenient course, remembering that this speed must be $k(\theta - \alpha)$.

If $\alpha = 30°$, $v = 6$, and $k = 8$, equation (3) gives about 45° for θ.

We shall get to windward, then, under sail and gas combined, most rapidly if we keep the boat about four points off the wind. Can we perhaps do better by dropping sail and going dead to windward under gas alone? To get a mile to windward under sail and motor will take us $(\sec \theta)/[v + k(\theta - \alpha)]$ hours. Computing this value for $\theta = 45°$, $\alpha = 30°$, $v = 6$, and $k = 8$, we get 0.175 hour. To get a mile to windward under gas takes 0.167 hour.

The time required to get a mile to windward under gas on a course making the angle β with the wind is $(\sec \beta)/v$. This is less than 0.175 if β is less than 18°.

Summing up, we see that if short-handed our "sailing directions" should read as follows:

(*a*) For any goal lying in the angle whose vertex is at the boat and whose sides are inclined 18° to the wind's direction, drop sail and head for the goal under gas alone.

(*b*) For any goal between these lines and lines through the boat and inclined 45° to the wind's direction, tack under sail and gas, keeping four points off the wind on each tack.

(*c*) For all other goals, go straight under sail and gas.

For a second and much closer approximation, let us assume that the velocity under sail is $a(\theta - \alpha) + b(\theta - \alpha)^2$. In practice a and b can be determined from the observed speed on two courses, preferably chosen so that the angle $\theta - \alpha$ is small.

For sail alone $F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2$, and equation (1) becomes $[a(\theta - \alpha) + b(\theta - \alpha)^2] \tan \theta = a + 2b(\theta - \alpha)$ or

$$\left(\tan \theta - \frac{2b}{a}\right)(\theta - \alpha) + \frac{b}{a}(\theta - \alpha)^2 \tan \theta = 1. \tag{4}$$

For an example, suppose as before that $\alpha = 30°$, and suppose that the boat it found to make 2 knots when $\theta - \alpha$ is $\frac{1}{4}$ and 3 knots when $\theta - \alpha$ is $\frac{1}{2}$.

We have

$$2 = \frac{a}{4} + \frac{b}{16}; \qquad 3 = \frac{a}{2} + \frac{b}{4}.$$

Whence $a = 10, b = -8, b/a = -0.8$. Equation (4) gives $(\tan\theta + 1.6)(\theta - \alpha) - 0.8(\theta - \alpha)^2 \tan\theta = 1$, and

$$\theta = 55° \text{ approximately.}$$

If we use sail and motor, $F(\theta) = v + a(\theta - \alpha) + b(\theta - \alpha)^2$. Equation (3) becomes

$$[v + a(\theta - \alpha) + b(\theta - \alpha)^2] \tan\theta = a + 2b(\theta - \alpha)$$

or

$$\left(\tan\theta - \frac{2b}{a}\right)(\theta - \alpha) + \left[\frac{b}{a}(\theta - \alpha)^2 + \frac{v}{a}\right]\tan\theta = 1. \tag{5}$$

For our example, equation (5) becomes

$$(\tan\theta + 1.6)(\theta - \alpha) + [0.6 - 0.8(\theta - \alpha)^2]\tan\theta = 1,$$

$$\theta = 41°.5 \text{ approximately.}$$

The best time for a mile to windward under sail and motor is

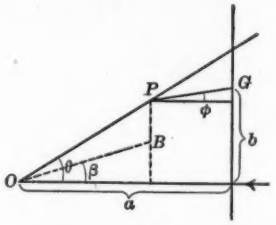
$$(\sec\theta)/[v + a(\theta - \alpha) + b(\theta - \alpha)^2],$$

or about 0.1738 hour. Under motor alone, in this time, the boat would go 1.043 miles, and $\sec^{-1} 1.043$ is about 16°.5.

Our sailing directions would be the same as in our first example, except that in (a) 18° would be changed to 16°.5, and in (b) 45° would be changed to 41°.5.

Hitherto, we have supposed the crew short-handed, so that sail was neither raised nor lowered in transit.

In a light wind where β is an angle of appreciable magnitude and we are aiming at a goal in the sector $\theta - \beta$, we can do best by using sail and motor



part way and motor only the remainder of the way. Let us consider this new problem. Let θ and β have their old signification and let φ be the angle which

the course makes with the wind after we lower sail. Let P be the turning point and G be the goal. Then, since the whole time of transit is the time that it would take to go the distance $OB + PG$ under motor, we must choose φ so that this distance shall be a minimum.

$$PG = \frac{(a \tan \theta - b) \cos \theta}{\sin (\theta - \varphi)}, \qquad OB = (a - PG \cos \varphi) \sec \beta.$$

$$OB + PG = a \sec \beta - PG(\cos \varphi \sec \beta - 1).$$

This will be a minimum when $PG \cos \varphi(\sec \beta - \sec \varphi)$ is a maximum; and that will be the case when u is a maximum if

$$u = \frac{\cos \theta \cos \varphi}{\sin (\theta - \varphi)} [\sec \beta - \sec \varphi] = \frac{\sec \beta - \sec \varphi}{\tan \theta - \tan \varphi}.$$

$$\frac{du}{d\varphi} = \frac{(\tan \varphi - \tan \theta) \sec \varphi \tan \varphi + (\sec \beta - \sec \varphi) \sec^2 \varphi}{(\tan \theta - \tan \varphi)^2} = 0.$$

$$(\tan \theta - \tan \varphi) \tan \varphi = (\sec \beta - \sec \varphi) \sec \varphi.$$

$$\tan \theta \tan \varphi + 1 = \sec \beta \sec \varphi.$$

$$\cos \varphi + \tan \theta \sin \varphi = \sec \beta. \tag{6}$$

φ determined from equation (6) gives the bearing of the goal when the proper turning point is reached.

If $\theta = 45°$ and $\beta = 18°$, $\varphi = 3°$. If $\theta = 41°.5$ and $\beta = 16°.5$, $\varphi = 3°-$.

If the yachtsman has patience and a soul that longs for accuracy he can amuse himself by trying a third approximation for $F(\theta)$, namely,

$$F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2 + c(\theta - \alpha)^3.$$

THE ACCELERATIONS OF THE POINTS OF A RIGID BODY.

By PETER FIELD AND ALEXANDER ZIWET, University of Michigan.

INTRODUCTION.

It has long been known that, in plane motion, the acceleration field is completely determined by the accelerations of any two points; and the point of zero acceleration, or "acceleration center," for such a motion is discussed in most works on mechanics.[1] The corresponding problems for three dimensions, that is, the determination of the acceleration field from the accelerations of three points and the construction of the acceleration center (or centers), are less widely known. Vector methods are particularly appropriate for their solution. They

[1] Compare AMERICAN MATHEMATICAL MONTHLY, Vol. XXI, 1914, pp. 105–113.

bring out very clearly to what extent the instantaneous motion remains indeterminate when the accelerations of three points are given, and lead to the enumeration of the various cases that may arise concerning the existence of points of zero acceleration. The dynamical interpretation of the result of Art. 20 would also appear to be noteworthy.

For the determination of the central axis from the accelerations of three points compare J. PETERSEN, *Kinematik*, 1884, Art. 36, pp. 47–49, and R. MEHMKE, in *Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt*, p. 77.

I. VELOCITY.

1. Let O, P, Q be any three non-collinear points of a rigid body and put

$$P - O = \boldsymbol{p}, \quad Q - O = \boldsymbol{q}$$

The assumption that O, P, Q are different and not collinear gives

$$\boldsymbol{p} \wedge \boldsymbol{q} \neq 0.$$

The assumption of rigidity gives

$$\boldsymbol{p}^2 = \text{const.}, \quad \boldsymbol{q}^2 = \text{const.}, \quad (\boldsymbol{q} - \boldsymbol{p})^2 = \text{const.}$$

Differentiating with respect to the time we find:

$$(1) \qquad \dot{\boldsymbol{p}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \quad \dot{\boldsymbol{p}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{p} = 0,$$

where $\dot{\boldsymbol{p}}$, $\dot{\boldsymbol{q}}$ are the velocities of P, Q, *relative to* O.

If R be any fourth point of the body and we put $R - O = \boldsymbol{r}$ we have the additional rigidity conditions

$$\boldsymbol{r}^2 = \text{const.}, \quad (\boldsymbol{r} - \boldsymbol{p})^2 = \text{const.}, \quad (\boldsymbol{r} - \boldsymbol{q})^2 = \text{const.},$$

whence

$$(2) \qquad \dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} + \dot{\boldsymbol{p}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{r} = 0.$$

We proceed to prove that *the velocities* $\dot{O}$, $\dot{P}$, $\dot{Q}$ *of any three non-collinear points* O, P, Q *of a rigid body completely determine the velocities of all points of the body.*

2. Suppose first that $\dot{\boldsymbol{p}} = 0$ and $\dot{\boldsymbol{q}} = 0$ so that O, P, Q have equal velocities. Then by (2)

$$\dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} = 0.$$

Now if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} \neq 0$ so that $\boldsymbol{r}$, $\boldsymbol{p}$, $\boldsymbol{q}$ can be taken as reference vectors for $\dot{\boldsymbol{r}}$ the equations show that $\dot{\boldsymbol{r}} = 0$. If, however, $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} = 0$ we can put $\boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q}$, whence $\dot{\boldsymbol{r}} = a\dot{\boldsymbol{p}} + b\dot{\boldsymbol{q}} = 0$, as before. Hence *if any three non-collinear points have equal velocities the velocities of all points are equal.* The instantaneous motion in this case is called a **translation**.

3. Suppose next that $\dot{\boldsymbol{p}} = 0$, $\dot{\boldsymbol{q}} \neq 0$. The conditions (1) and (2) reduce to

$$\dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \quad \dot{\boldsymbol{q}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{r} = 0.$$

As by the first two conditions $\dot{\boldsymbol{q}}$ is normal to both $\boldsymbol{q}$ and $\boldsymbol{p}$ we can put $\dot{\boldsymbol{q}} = k\boldsymbol{p} \wedge \boldsymbol{q}$. The last equation then becomes $\dot{\boldsymbol{r}} \times \boldsymbol{q} + k\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{r} = 0$, *i. e.*, $(\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{q} = 0$; the last three equations can therefore be written in the form

$$(\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{r} = 0, \qquad (\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{p} = 0, \qquad (\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{q} = 0;$$

hence, if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} \neq 0$:

$$\dot{\boldsymbol{r}} = k\boldsymbol{p} \wedge \boldsymbol{r};$$

if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} = 0$ we can again put $\boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q}$, whence $\dot{\boldsymbol{r}} = b\dot{\boldsymbol{q}} = bk\boldsymbol{p} \wedge \boldsymbol{q}$ $= k\boldsymbol{p} \wedge (\boldsymbol{r} - a\boldsymbol{p}) = k\boldsymbol{p} \wedge \boldsymbol{r}$. Thus, if $O = P \neq Q$, there exists a vector $\boldsymbol{u} = k\boldsymbol{p}$, parallel to OP, such that the velocity of every point R can be derived from it by the formula

$$\dot{\boldsymbol{r}} = \boldsymbol{u} \wedge \boldsymbol{r}.$$

The scalar k is determined by observing that $\dot{\boldsymbol{q}} = k\boldsymbol{p} \wedge \boldsymbol{q}$ whence

$$k = \frac{\operatorname{mod} \dot{\boldsymbol{q}}}{\operatorname{mod} \boldsymbol{p} \cdot \operatorname{mod} \boldsymbol{q} \cdot \sin (\boldsymbol{p}, \boldsymbol{q})}.$$

4. Finally, suppose that $\dot{\boldsymbol{p}} \neq 0$ and $\dot{\boldsymbol{q}} \neq 0$. It can be shown that in this case, too, there exists a vector $\boldsymbol{w}$ such that

$$\dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r},$$

whatever the point R. For, as the vector $\boldsymbol{w}$ is to satisfy the conditions

$$\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}, \qquad \dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q},$$

it must be normal to both $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$; it must therefore be of the form

$$\boldsymbol{w} = k\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}.$$

To determine k we have only to substitute this value of $\boldsymbol{w}$ in the two preceding equations. Owing to the first two of (1) we find

$$\dot{\boldsymbol{p}} = k(\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}) \wedge \boldsymbol{p} = -k\dot{\boldsymbol{q}} \times \boldsymbol{p} \cdot \dot{\boldsymbol{p}},$$

$$\dot{\boldsymbol{q}} = k(\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}) \wedge \boldsymbol{q} = \quad k\dot{\boldsymbol{p}} \times \boldsymbol{q} \cdot \dot{\boldsymbol{q}};$$

and owing to the third of (1) these equations give the same value for k, viz.,

$$k = -\frac{1}{\dot{\boldsymbol{q}} \times \boldsymbol{p}} = \frac{1}{\dot{\boldsymbol{p}} \times \boldsymbol{q}}.$$

The vector

$$\boldsymbol{w} = -\frac{\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}}{\dot{\boldsymbol{q}} \times \boldsymbol{p}} = \frac{\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}}{\dot{\boldsymbol{p}} \times \boldsymbol{q}} \tag{3}$$

gives therefore $\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q}$. That it also gives

$$\dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r}, \tag{4}$$

whatever R, appears by putting $R - O = r = ap + bq + cp \wedge q$, whence

$$\begin{aligned}\dot{r} &= a\dot{p} + b\dot{q} + c(\dot{p} \wedge q + p \wedge \dot{q}) \\ &= aw \wedge p + bw \wedge q + c[(w \wedge p) \wedge q + p \wedge (w \wedge q)] \\ &= aw \wedge p + bw \wedge q + cw \wedge (p \wedge q) \\ &= w \wedge (ap + bq + cp \wedge q) = w \wedge r.\end{aligned}$$

It is thus proved that *the velocities of all points of the body can be found from those of any three non-collinear points.*

5. The velocities $\dot{R}$, $\dot{S}$ of two points R, S are equal if $\dot{r} = \dot{s}$, *i. e.*, if $w \wedge r = w \wedge s$, whence $w \wedge (r - s) = 0$. It follows that all points of any line parallel to w have equal velocities. This shows that the case of Art. 3, where two of the given velocities were assumed equal, is no less general than that of Art. 4; but the case of Art. 3 suggests more directly the introduction of the auxiliary vectors u and w.

In general, there is no point of zero velocity. But if such a point exists, say O, then all points of the line l through O, parallel to w, have zero velocity, and the velocity $\dot{R} = \dot{r} = w \wedge r$ of every other point R is normal to w and r and in magnitude proportional to the distance of R from l. The instantaneous motion in this case is called a **rotation**; the line l is called the **axis**, the vector w the **angular velocity**, of the rotation.

6. Even in the general case the vector w may be called the angular velocity. The instantaneous motion is completely determined by the angular velocity w and the linear velocity $\dot{O}$ of any one point. For then the velocity of every point R is, by (4):

$$\dot{R} = \dot{O} + w \wedge r.$$

II. Acceleration.

7. Differentiating the equations $\dot{p} = w \wedge p$ and $\dot{q} = w \wedge q$ we find:

$$\begin{aligned}\ddot{p} &= \dot{w} \wedge p + w \wedge \dot{p} = \dot{w} \wedge p + w \wedge (w \wedge p), \\ \ddot{q} &= \dot{w} \wedge q + w \wedge \dot{q} = \dot{w} \wedge q + w \wedge (w \wedge q).\end{aligned} \tag{5}$$

These equations are not linear in w. Eliminating $\dot{w}$ we have:

$$\ddot{p} \times p = w \wedge (w \wedge p) \times p = (w \times p)^2 - w^2 p^2,$$

$$\ddot{q} \times q = w \wedge (w \wedge q) \times q = (w \times q)^2 - w^2 q^2, \tag{5'}$$

$$\ddot{p} \times q + \ddot{q} \times p = 2(w \times p \cdot w \times q - p \times q \cdot w^2).$$

If in the trihedral formed by p, q, w at O (Fig. 1) we denote the face angles (p, q), (p, w), (q, w) by a, b, c and the dihedral angle at w by α, and if we put

for the sake of brevity

$$-\frac{\ddot{\boldsymbol{p}}\times\boldsymbol{p}}{\boldsymbol{p}^2}=B,\qquad -\frac{\ddot{\boldsymbol{q}}\times\boldsymbol{q}}{\boldsymbol{q}^2}=C,$$

$$-\frac{\ddot{\boldsymbol{p}}\times\boldsymbol{q}+\ddot{\boldsymbol{q}}\times\boldsymbol{p}}{2\,\mathrm{mod}\,\boldsymbol{p}\cdot\mathrm{mod}\,\boldsymbol{q}}=A,$$

the equations become

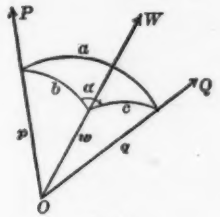


Fig. 1.

(6) $\boldsymbol{w}^2\sin^2 b=B,\quad \boldsymbol{w}^2\sin^2 c=C,\quad \boldsymbol{w}^2\sin b\sin c\cos\alpha=A,$

whence

$$\cos\alpha=\frac{A}{\sqrt{BC}}. \tag{7}$$

When the accelerations of the three points O, P, Q are given, the quantities A, B, C can be found. But the vector $\boldsymbol{w}$ is not uniquely determined: its extremity W (Fig. 1) may lie on either side of the plane OPQ, and the sense of $\boldsymbol{w}$ remains indeterminate. Thus there are four distinct solutions.

Moreover, the accelerations $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$ cannot be arbitrarily prescribed. To obtain a real $\boldsymbol{w}$ it is necessary that $A^2\leqq BC$, *i. e.*,

$$(\ddot{\boldsymbol{p}}\times\boldsymbol{q}+\ddot{\boldsymbol{q}}\times\boldsymbol{p})^2\leqq 4\ddot{\boldsymbol{p}}\times\boldsymbol{p}\cdot\ddot{\boldsymbol{q}}\times\boldsymbol{q}.$$

8. Suppose, in particular, that the points O, P, Q are such that $\boldsymbol{p}$ and $\boldsymbol{q}$ are rectangular unit vectors and put

$$\boldsymbol{w}=w_1\boldsymbol{p}+w_2\boldsymbol{q}+w_3\boldsymbol{p}\wedge\boldsymbol{q},$$

so that w_1, w_2, w_3 are the rectangular coordinates of $\boldsymbol{w}$ with respect to $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}\wedge\boldsymbol{q}$. If, moreover, p_1, p_2, p_3 and q_1, q_2, q_3 are the coordinates of $\ddot{\boldsymbol{p}}$ and $\ddot{\boldsymbol{q}}$ with respect to the same reference set, the equations (5′) become

$$w_2^2+w_3^2=-p_1,\quad w_3^2+w_1^2=-q_2,\quad 2w_1w_2=p_2+q_1. \tag{8}$$

These equations give

$$\begin{aligned} w_1^2 &= \tfrac{1}{2}[p_1-q_2+\sqrt{(p_1-q_2)^2+(p_2+q_1)^2}],\\ w_2^2 &= -\tfrac{1}{2}[p_1-q_2-\sqrt{(p_1-q_2)^2+(p_2+q_1)^2}], \end{aligned} \tag{9}$$

where the sign of the square root has been selected so as to make w_1 and w_2 real. For w_3 we find:

$$w_3^2=-p_1-w_2^2=-q_2-w_1^2=-\tfrac{1}{2}[p_1+q_2+\sqrt{(p_1-q_2)^2+(p_2+q_1)^2}].$$

Hence w_3 is real only if

$$p_1q_2\geqq\left(\frac{p_2+q_1}{2}\right)^2,$$

which is the condition at the end of Art. 7.

9. *To express* $\dot{\boldsymbol{w}}$ *in terms of* $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$, *and* $\boldsymbol{w}$ we may either differentiate (3) or observe that, referring $\dot{\boldsymbol{w}}$ to $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{w}$, we must have:

$$\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \dot{\boldsymbol{w}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \boldsymbol{p} + \boldsymbol{p} \wedge \dot{\boldsymbol{w}} \times \boldsymbol{w} \cdot \boldsymbol{q} + \boldsymbol{p} \wedge \boldsymbol{q} \times \dot{\boldsymbol{w}} \cdot \boldsymbol{w}$$

Now by (5)

$$\dot{\boldsymbol{w}} \wedge \boldsymbol{q} \times \boldsymbol{w} = \ddot{\boldsymbol{q}} \times \boldsymbol{w}, \qquad \boldsymbol{p} \wedge \dot{\boldsymbol{w}} \times \boldsymbol{w} = -\ddot{\boldsymbol{p}} \times \boldsymbol{w},$$

$$\boldsymbol{p} \wedge \boldsymbol{q} \times \dot{\boldsymbol{w}} = \boldsymbol{p} \times \boldsymbol{q} \wedge \dot{\boldsymbol{w}} = -\boldsymbol{p} \times \ddot{\boldsymbol{q}} + \boldsymbol{p} \times \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q})$$

$$= -\boldsymbol{p} \times \ddot{\boldsymbol{q}} + \boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{w}^2 \cdot \boldsymbol{p} \times \boldsymbol{q};$$

hence

$$\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \dot{\boldsymbol{w}} = \boldsymbol{w} \times \ddot{\boldsymbol{q}} \cdot \boldsymbol{p} - \boldsymbol{w} \times \ddot{\boldsymbol{p}} \cdot \boldsymbol{q}$$

(10)

$$+ (\boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{w}^2 \cdot \boldsymbol{p} \times \boldsymbol{q} - \boldsymbol{p} \times \ddot{\boldsymbol{q}})\boldsymbol{w}$$

The vector $\dot{\boldsymbol{w}}$ may be called the *angular acceleration.*

10. *Acceleration of any point* R. Putting, as in Art. 4,

$$R - O = \boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q},$$

we have

$$\ddot{\boldsymbol{r}} = a\ddot{\boldsymbol{p}} + b\ddot{\boldsymbol{q}} + c(\ddot{\boldsymbol{p}} \wedge \boldsymbol{q} + \boldsymbol{p} \wedge \ddot{\boldsymbol{q}} + 2\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}).$$

It may be observed that the acceleration of every point of the plane OPQ is given by

$$\ddot{\boldsymbol{r}} = a\ddot{\boldsymbol{p}} + b\ddot{\boldsymbol{q}};$$

i. e., it is determined completely by the accelerations $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$, of any three non-collinear points of this plane. But the acceleration of any point R not in this plane depends in addition, on the vector $\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}} = \dot{\boldsymbol{p}} \times \boldsymbol{q} \cdot \boldsymbol{w}$.

On the other hand, the relation (4) gives

$$\ddot{\boldsymbol{r}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}); \tag{11}$$

i. e., the acceleration $\ddot{R}$ of every point R is determined by the vectors $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$ and the acceleration $\ddot{O}$ of any one point O.

III. Center of Acceleration.

11. To see whether a point R of zero acceleration (or a *center of acceleration*) exists, let us first assume that $\boldsymbol{w} \times \dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \neq 0$, *i. e.*, $(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 \neq 0$, so that we can put

$$R - O = \boldsymbol{r} = a\boldsymbol{w} + b\dot{\boldsymbol{w}} + c\boldsymbol{w} \wedge \dot{\boldsymbol{w}}.$$

We then have to determine a, b, c so as to make $\ddot{R} = 0$, *i. e.*, by (11),

$$\ddot{O} + \dot{\boldsymbol{w}} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = 0.$$

Substituting for $\boldsymbol{r}$ its value we obtain the condition:

$$\ddot{O} + a\dot{\boldsymbol{w}} \wedge \boldsymbol{w} + c\dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) + b\boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) - c\boldsymbol{w}^2 \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0.$$

Multiplying by $\times \boldsymbol{w}$, $\times \dot{\boldsymbol{w}}$, $\times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})$ we find for a, b, c the conditions:

$$\ddot{O} \times \boldsymbol{w} + c\dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \times \boldsymbol{w} = 0,$$

$$\ddot{O} \times \dot{\boldsymbol{w}} + b\boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \times \dot{\boldsymbol{w}} = 0,$$

$$\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} - a(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 - c\boldsymbol{w}^2 \cdot (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 = 0,$$

whence

$$a = \frac{\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w}^2}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}, \qquad b = \frac{\ddot{O} \times \dot{\boldsymbol{w}}}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}, \qquad c = -\frac{\ddot{O} \times \boldsymbol{w}}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}.$$

The acceleration center is therefore given by the equation

$$(12) \quad (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 \boldsymbol{r} = \ddot{O} \times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \boldsymbol{w}^2 \cdot \boldsymbol{w}) \cdot \boldsymbol{w} + \ddot{O} \times \dot{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}.$$

This equation shows that whenever $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} \neq 0$ there exists one and only one acceleration center; if in particular $\ddot{O} = 0$, O is the center.

12. It remains to discuss the cases when $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. We first assume $\ddot{O} \neq 0$.

(*a*) Suppose $\ddot{O} \neq 0$, $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. We then have $\dot{\boldsymbol{w}} = k\boldsymbol{w}$, where $k \neq 0$, and the condition for R to have zero acceleration becomes

$$\ddot{O} + k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = 0.$$

Multiplying by $\boldsymbol{w} \times$ we find $\boldsymbol{w} \times \ddot{O} = 0$ as a first necessary condition for the existence of an acceleration center in this case.

It follows that $\boldsymbol{w} \wedge \ddot{O} \neq 0$ so that we can put $\boldsymbol{r} = a\boldsymbol{w} + b\ddot{O} + c\boldsymbol{w} \wedge \ddot{O}$; substituting this value we find if $\boldsymbol{w} \times \ddot{O} = 0$:

$$[1 - (kc + b)\boldsymbol{w}^2]\ddot{O} + (bk - c\boldsymbol{w}^2)\boldsymbol{w} \wedge \ddot{O} = 0,$$

whence

$$(kc + b)\boldsymbol{w}^2 = 1, \qquad kb = c\boldsymbol{w}^2,$$

i. e.,

$$b = \frac{1}{k^2 + \boldsymbol{w}^2}, \qquad c = \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)},$$

while a remains indeterminate. Thus, *if* $\ddot{O} \neq 0$, $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, *while* $\dot{\boldsymbol{w}}$ *is parallel to* $\boldsymbol{w}$, *there is no point of zero acceleration unless* $\ddot{O}$ *is normal to* $\boldsymbol{w}$; *if* $\dot{\boldsymbol{w}} \times \ddot{O} = 0$, *every point of the line*

$$R = O + a\boldsymbol{w} + \frac{1}{k^2 + \boldsymbol{w}^2}\ddot{O} + \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)}\boldsymbol{w} \wedge \ddot{O},$$

and no other point, has zero acceleration.

(*b*) *Under the same conditions, except that* $\dot{\boldsymbol{w}} = 0$, *there is no acceleration center if* $\boldsymbol{w} \times \ddot{O} \neq 0$; *if* $\boldsymbol{w} \times \ddot{O} = 0$, *the points of the line*

$$R = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\ddot{O}$$

have zero acceleration.

(*c*) If $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$ we have $\ddot{R} = \ddot{O} + \dot{\boldsymbol{w}} \wedge \boldsymbol{r}$; hence a first condition is $\dot{\boldsymbol{w}} \times \ddot{O} = 0$. We can therefore put $\boldsymbol{r} = a\dot{\boldsymbol{w}} + b\ddot{O} + c\dot{\boldsymbol{w}} \wedge \ddot{O}$ so that

$$\ddot{R} = \ddot{O} + b\dot{\boldsymbol{w}} \wedge \ddot{O} - c\dot{\boldsymbol{w}}^2 \cdot \ddot{O} = 0;$$

hence

$$b = 0, \qquad c = \frac{1}{\dot{\boldsymbol{w}}^2},$$

while a is arbitrary. Thus, *if* $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$ *there is no acceleration center if* $\dot{\boldsymbol{w}} \times \ddot{O} \neq 0$*; if* $\dot{\boldsymbol{w}}$ *is normal to* $\ddot{O}$ *all points of the line*

$$R = O + a\dot{\boldsymbol{w}} + \frac{1}{\dot{\boldsymbol{w}}^2}\dot{\boldsymbol{w}} \wedge \ddot{O}$$

have zero acceleration.

(*d*) *If* $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} = 0$ *there is evidently no point of zero acceleration.*

13. Next suppose $\ddot{O} = 0$ and $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. Then we have the following cases:

(*e*) $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$. As $\dot{\boldsymbol{w}} = k\boldsymbol{w}$, where $k \neq 0$, the condition for R to have zero acceleration is

$$0 = k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \times \boldsymbol{r} \cdot \boldsymbol{w} - \boldsymbol{w}^2 \cdot \boldsymbol{r}.$$

Hence in this case *every point of the line through* O *parallel to* $\boldsymbol{w}$ *has zero acceleration.*

(*f*) $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} = 0$. The result is evidently the same.

(*g*) $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$. As $\ddot{R} = \dot{\boldsymbol{w}} \wedge \boldsymbol{r}$, *every point of the line through* O *parallel to* $\dot{\boldsymbol{w}}$ *has zero acceleration.*

(*h*) $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} = 0$. As $\ddot{R} = 0$, *every point has zero acceleration.*

IV. Applications.

14. *Example I: Projectile whose axis of spin is permanent.* Let O be the centroid; as $\dot{\boldsymbol{w}} = 0$ we have case (*b*) of Art. 12. Hence there is no point of zero acceleration unless $\boldsymbol{w} \times \ddot{O} = 0$.

Suppose this condition satisfied. Let O_0 be the initial position of the centroid; $\boldsymbol{j}_0$, $\boldsymbol{k}_0$ unit vectors, $\boldsymbol{j}_0$ horizontal and at right angles to $\boldsymbol{w}$, $\boldsymbol{k}_0$ vertical downwards. Then, if v_1, v_2, v_3 are the coördinates of the velocity of O along $\boldsymbol{w}$, $\boldsymbol{j}_0$, $\boldsymbol{k}_0$, the position of the centroid at the time t is $O = O_0 + v_1 t\boldsymbol{w} + v_2 t\boldsymbol{j}_0 + (v_3 t + \frac{1}{2}gt^2)\boldsymbol{k}_0$; hence

$$R = O_0 + v_1 t\boldsymbol{w} + v_2 t\boldsymbol{j}_0 + (v_3 t + \tfrac{1}{2}gt^2)\boldsymbol{k}_0 + a\boldsymbol{w} + \frac{g}{\boldsymbol{w}^2}\boldsymbol{k}_0.$$

This is, for a constant t, the line whose points have zero accelerations. For t as a variable parameter, the same equation represents the ruled surface whose generators are those lines of space which in the course of time come to have this property. Thus, the locus in space of all those points that have zero acceleration is a parabolic cylinder.

To obtain the locus, in the body, of these lines, let $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors fixed

in the body and initially coinciding with $\boldsymbol{j}_0, \boldsymbol{k}_0$. Then we have for any point S of the body:

$$S = O + a\boldsymbol{w} + \frac{1}{w^2}(\boldsymbol{g} \times \boldsymbol{j} \cdot \boldsymbol{j} + \boldsymbol{g} \times \boldsymbol{k} \cdot \boldsymbol{k})$$

$$= O + a\boldsymbol{w} + \frac{g}{w^2} \sin wt \cdot \boldsymbol{j} + \frac{g}{w^2} \cos wt \cdot \boldsymbol{k},$$

where $g = \operatorname{mod} \boldsymbol{g}$, $w = \operatorname{mod} \boldsymbol{w}$. The surface is a circular cylinder.

15. *Example II*: *Homogeneous circular disk rolling down an inclined plane, starting from rest.*

Let O be the position of the centroid at the time t. We have $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$; *i. e.*, we have case (*a*) of Art. 12, with $\boldsymbol{w} \times \ddot{O} = 0$. Hence, if O_0 is the initial position of the centroid, the points of zero acceleration are given by the equation

$$R = O_0 + (O - O_0) + a\boldsymbol{w} + \frac{1}{k^2 + w^2}\ddot{O} + \frac{k}{w^2(k^2 + w^2)}\boldsymbol{w} \wedge \ddot{O}.$$

If we put

$$\ddot{O} = a_1\boldsymbol{u}_0, \quad O - O_0 = \tfrac{1}{2}a_1t^2\boldsymbol{u}_0, \quad \boldsymbol{w} = a_2t\boldsymbol{w}_0, \quad k = \frac{\operatorname{mod} \dot{\boldsymbol{w}}}{\operatorname{mod} \boldsymbol{w}} = \frac{1}{t}, \quad \boldsymbol{w}_0 \wedge \boldsymbol{u}_0 = \boldsymbol{v}_0,$$

the equation becomes

$$R = O_0 + \tfrac{1}{2}a_1t^2\boldsymbol{u}_0 + aa_2t\boldsymbol{w}_0 + \frac{t^2}{1 + a_2^2t^4}a_1\boldsymbol{u}_0 + \frac{a_1}{a_2(1 + a_2^2t^4)}\boldsymbol{v}_0$$

$$= O_0 + aa_2t\boldsymbol{w}_0 + \frac{3 + a_2t^4}{2(1 + a_2^2t^4)}a_1t^2\boldsymbol{u}_0 + \frac{a_1}{a_2(1 + a_2^2t^4)}\boldsymbol{v}_0.$$

This is the locus, in space, of the points of zero acceleration. To find the locus in the body let $\boldsymbol{u}, \boldsymbol{v}$ be unit vectors fixed in the body and initially coinciding with $\boldsymbol{u}_0, \boldsymbol{v}_0$. The angle made by $\boldsymbol{u}$ with $\boldsymbol{u}_0$ is $\frac{1}{2}a_2t^2$, and the equation of the locus is

$$S = O + aa_2t\boldsymbol{w}_0 + \frac{t^2}{1 + a_2t^4}(a_1\boldsymbol{u}_0 \times \boldsymbol{u} \cdot \boldsymbol{u} + a_1\boldsymbol{u}_0 \times \boldsymbol{v} \cdot \boldsymbol{v})$$

$$+ \frac{a_1}{a_2(1 + a_2^2t^4)}(\boldsymbol{v}_0 \times \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v}_0 \times \boldsymbol{v} \cdot \boldsymbol{v}),$$

where

$$\boldsymbol{u}_0 \times \boldsymbol{u} = \cos \tfrac{1}{2}a_2t^2, \qquad \boldsymbol{u}_0 \times \boldsymbol{v} = -\sin \tfrac{1}{2}a_2t^2, \qquad \boldsymbol{v}_0 \times \boldsymbol{u} = \sin \tfrac{1}{2}a_2t^2,$$

$$\boldsymbol{v}_0 \times \boldsymbol{v} = \cos \tfrac{1}{2}a_2t^2.$$

16. *Example III*: *Projectile whose momental ellipsoid at the centroid is an ellipsoid of revolution.*

This is the general case so that we have to use equation (12), viz.,

$$R = O + \frac{1}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}[(\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \boldsymbol{w}^2 \cdot \ddot{O} \times \boldsymbol{w})\boldsymbol{w} + \ddot{O} \times \dot{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}].$$

Taking for O the centroid (Fig. 2) we have $\ddot{O} = \boldsymbol{g}$, $\ddot{O} \times \dot{\boldsymbol{w}} = 0$, and $\boldsymbol{w} \times \dot{\boldsymbol{w}} = 0$; hence, denoting the moduli of $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$, $\boldsymbol{g}$ by w, $\dot{w}$, g, and the constant angle made by $\boldsymbol{w}$ with $\boldsymbol{g}$ by θ, we have:

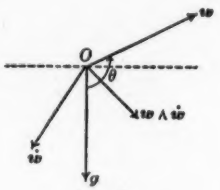


FIG. 2.

$$R = O + \frac{1}{w^2\dot{w}^2}[(gw\dot{w}\sin\theta + gw^3\cos\theta)\boldsymbol{w} - gw\cos\theta \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}].$$

This shows that the body locus is a circle while the space locus is a transcendental curve.

CENTRAL AXIS.

17. At any given instant t, the locus of those points of the body whose velocity is parallel to $\boldsymbol{w}$ is called the *instantaneous screw axis* of the motion, or the *central axis* of the velocity field. Assuming $\boldsymbol{w} \neq 0$, *i. e.*, assuming that the instantaneous motion is not a translation, it is readily shown that this locus is a straight line parallel to $\boldsymbol{w}$.

For, if the velocity $\dot{R} = \dot{O} + \boldsymbol{w} \wedge \boldsymbol{r}$ of any point R of the body is to be parallel to $\boldsymbol{w}$ we must have $\dot{R} \wedge \boldsymbol{w} = 0$, *i. e.*,

$$\dot{O} \wedge \boldsymbol{w} + (\boldsymbol{w} \wedge \boldsymbol{r}) \wedge \boldsymbol{w} = 0.$$

Putting $\boldsymbol{r} = a\boldsymbol{w} + b\dot{O} + c\boldsymbol{w} \wedge \dot{O}$ we find

$$\dot{O} \wedge \boldsymbol{w} + [\boldsymbol{w} \wedge (b\dot{O} + c\boldsymbol{w} \wedge \dot{O})] \wedge \boldsymbol{w} = 0,$$

i. e.,

$$\dot{O} \wedge \boldsymbol{w} + b(\boldsymbol{w} \wedge \dot{O}) \wedge \boldsymbol{w} + c\boldsymbol{w}^2 \cdot \boldsymbol{w} \wedge \dot{O} = 0.$$

Hence $b = 0$, $c = 1/\boldsymbol{w}^2$, while a remains arbitrary. The equation of the central axis at the time t is therefore

$$R = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}. \tag{13}$$

Every value of a gives a point R of the central axis; if R_0 corresponds to $a = 0$, we have $R - R_0 = a\boldsymbol{w}$; *i. e.*, the central axis is parallel to $\boldsymbol{w}$.

18. If in (13) the point O and the vectors $\boldsymbol{w}$ and $\dot{O}$ are given as functions of the time t, so that R becomes a function of the parameters a and t, this equation (13) represents the ruled surface formed by those lines of the body which in the course of time become central axes.

To obtain the equation of the ruled surface formed by those lines of space

which in the course of time become central axes, let O_1 be a point fixed in space and $\dot{O}$ the velocity of that point of the body which at the time t coincides with O_1. Then the required equation is

$$R = O_1 + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}. \tag{14}$$

19. By (14), the velocity of any point R of the central axis is

$$\dot{R} = a\dot{\boldsymbol{w}} + \frac{d}{dt}\left(\frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}\right).$$

The ruled surface (14) is developable, *i. e.*, the motion of the central axis is a pure rotation, if the velocity $\dot{R}$ is parallel to the plane of the vectors $\boldsymbol{w}$ and $\dot{\boldsymbol{w}}$, *i. e.*, if $\dot{R} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. The condition for the surface (14) to be developable is therefore

$$\left(\frac{d}{dt}\frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}\right) \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0,$$

i. e.,

$$\left(-\frac{2\boldsymbol{w} \times \dot{\boldsymbol{w}}}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O} + \dot{\boldsymbol{w}} \wedge \dot{O} + \boldsymbol{w} \wedge \ddot{O}\right) \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0. \tag{15}$$

20. This equation contains $\ddot{O}$ only in the combination $\ddot{O} \times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \wedge \boldsymbol{w}$; *i. e.*, if $\ddot{O}$ be resolved along the rectangular vectors $\boldsymbol{w}$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}}$, and $\boldsymbol{u} = \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})$, the equation contains only the component of $\ddot{O}$ along $\boldsymbol{u}$.

This fact has an interesting dynamical interpretation. Let O be the centroid, and let $\boldsymbol{R}$, $\boldsymbol{H}$ be the vectors of the resultant force and couple for O. Then our result means that, if the surface of the screw axes is to be developable, $\boldsymbol{R}$ must have a fixed projection on $\boldsymbol{u}$; in other words, all values of $\boldsymbol{R}$ that give a developable surface are such that, when drawn from O, their extremities lie in a plane parallel to $\boldsymbol{w}$ and $\boldsymbol{w} \wedge \dot{\boldsymbol{w}}$.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated and provided with an introduction and notes by PHILIP E. B. JOURDAIN. The Open Court Publishing Company, Chicago, 1915. ix+211 pages.

The main purpose of this little volume, the first in the Open Court Series of Classics of Science and Philosophy, is to make accessible in English the two final papers of Georg Cantor on the theory of transfinite numbers, which embody the culmination of his researches on this subject, and which appeared in 1895 and 1897 in the *Mathematische Annalen* (Vols. 46 and 49). In order to give a proper setting to these two papers, the translator precedes them by an introduction of about 80 pages, in which he sketches briefly the progress of ideas which lead up to

these researches and results. A brief note collecting some of the chief developments which have been made in this theory since that time, brings the volume to a close.

In the introduction, we find some brief indications of the genesis of the modern conception of function, out of the consideration of the problem of the vibrating string and the theory of Fourier series. This is followed by a sketch of Weierstrass's theory of irrational numbers, in so far as Weierstrass probably exerted a most potent influence on Cantor. The main section of the introduction, however, is devoted to a résumé of the chief results obtained by Cantor in his researches preceding the papers translated. Beginning with his investigations in the theory of Fourier series, we are shown how Cantor was led to a closer study of point sets, on the one hand in the direction of their " Maechtigkeit " or power, and on the other to the consideration of their derivatives of various orders, the latter culminating in the notion of transfinite ordinal numbers.

The desirability of such an historical introduction is of course apparent. Aside from its value in preparing one for the papers which form the central portion of the book, it is always an interesting matter to trace the development of an idea, to go behind the scenes, so to speak, and see how a bit of investigation gradually assumes a final form. It is therefore very much to be regretted that our author in his attempts to be brief is not always perfectly clear. For instance his exposition of Weierstrass's theory of irrational numbers, on pages 18 and 19, would appeal to one who is not thoroughly conversant with the theory in question as a hopeless conglomeration of aggregates, sums, numerical quantities, and other elements. A simple illustration would probably have served the purpose much better than this attempt at generality without definiteness. Later on, in Section VII, page 52 and following, in the résumé of Cantor's 1883 Grundlagen, much would have been gained in clearness if the author had followed Cantor's original article a little more closely. In particular, the meaning of the word "Anzahl," which is translated enumeral, and seems to lead to confusion in the mind of the author, would have been perfectly clear at the outset as referring particularly to arrangement, rather than to " Maechtigkeit " or power. As it is, only after it has been used repeatedly and frequently do we really get any inkling as to the meaning to be assigned to the term enumeral. Of other sections which might well be improved, we might note the discussion of DuBois Reymond's work on point sets, on page 34; and the proof that the totality of real numbers is not enumerable on page 39.

To give the contents of the papers translated is hardly within the province of this review. Briefly, they contain the developments of the notion of cardinal number, that is, the power of a set, and the laws governing the usual algebraic processes as applied to them. A similar treatment is given for the ordinal numbers. The second of the two papers concerns itself mainly with well-ordered sets. The translation, considered as a whole, is fairly well done. Occasional passages to be sure are reminiscent of the German idiom, and perhaps too literal. For instance, the statement of Theorem *H* on page 147 is ambiguous as a result of sticking too closely to the German text.

In its general makeup and appearance, we think the book commendable. As a personal opinion, we feel that it would have tended towards clarity if the translator had followed Cantor in italicizing the theorems. Further, we find the printing of the page numbers of the original papers in the text in bold-faced type somewhat disconcerting. They might well have been relegated to the margin and a smaller size of type used. Of misprints, which are liable to cause ambiguity, we note: on page 147, line 14, D ought to be replaced by G; and in line 21, F ought to be replaced by f.

As a contribution to the history of transfinite numbers, and in making the main two papers on the foundations of these numbers accessible in English, we think this little volume is worthy of consideration. As an introduction to the subject, however, it cannot be unqualifiedly recommended.

T. H. HILDEBRANDT.

THE UNIVERSITY OF MICHIGAN.

Solid Geometry. By SOPHIA FOSTER RICHARDSON. Ginn and Co., Boston, 1914. v + 209 pages.

In applying to Professor Richardson's solid geometry the severe test of classroom use, it has been evident that the book possesses many excellent qualities. The use of a single letter to denote a line or a plane and the names skew lines, pencil of lines, pencil of planes, bundle of lines, bundle of planes have tended to make the work concise and the class demonstrations easy to follow. The first chapter contains more theorems than is usual in books on solid geometry. It provides a good drill in the more simple propositions that fix the concepts of perpendicularity and parallelism in space. A feature of interest to the student is the illustration of the principle of duality in several theorems differing only in the interchange of the words line and plane. Throughout the book there is an abundance of valuable and interesting exercises. However in many of the demonstrations a part of the work might well have been left for the student to supply.

The typographical errors are few, and in no instances misleading. The treatment of the incommensurable cases based on the Dedekind-Cantor theory of irrational numbers, presented in the appendix, is doubtless rigorous, but its advantages over the more ordinary methods have not been evident, since otherwise the work could be made much shorter and as rigorous as can be appreciated by first year college students. The propositions leading up to the determination of the surface and the volume of a sphere seem unnecessarily long and so involved that it becomes an arduous task even to quote them.

The formula for the volume of any parallelopiped is developed in two theorems, thus eliminating the celebrated "devil's coffin." This ruthless vandalism is regretted perhaps more by the expectant freshman, who has heard of its fame, than by the instructor, who has become calloused to its fearful and wonderful construction.

Taking everything into consideration it would seem that Professor Richardson

has written a book in advance of many text-books in solid geometry, and one adapted to students somewhat more mature than those of high school grade.

LENNIE P. COPELAND.

WELLESLEY COLLEGE,
WELLESLEY, MASS.

L'Opera "De Corporibus Regularibus" di Pietro Franceschi detto della Francesca usurpata da Fra Luca Pacioli (con dodici tavole). Memoria di G. Mancini. Reale Accademia dei Lincei (anno CCCXII, 1915), Serie Quinta, Volume XIV, Fascicolo VII[B]. Pp. 437–580; reprint, pp. 1–144.

During the renaissance there were two noted Italian artists who were also mathematicians. Both were in personal touch with the mathematical oracle of that day, Luca Pacioli. The two artists were Leonardo da Vinci and Pier della Francesca. Recently the mathematical achievements of both of these artists have been subjected to re-examination.[1] That Pacioli used a posthumous manuscript of Pier della Francesca without giving him due credit was affirmed by early writers on Pier della Francesca,[2] but denied by two later biographers of Pacioli.[3] This question was re-opened by G. Pittarelli[4] at the International Mathematical Congress held in Rome in 1908, who asserted that parts of Pacioli's *Divina proportione* (1509) were taken from a Vatican manuscript written by Pier della Francesca. This matter is re-investigated in the monograph under review, in which the Pier della Francesca manuscript, "De Corporibus Regularibus," is published, with extensive comments. While Pacioli's guilt seems now definitely established, it appears also from the history of the time that usurpations of this character were not uncommon. Thus, Pacioli's great compatriots Tartaglia and Cardan are both open to this charge. Gerolamo Mancini's monograph establishes Pier della Francesca's place in the history of the regular solids and also in the advancement of the theory of perspective. From the drawings it appears that Pier della Francesca was a skilled draftsman. His exposition of problems on areas is almost wholly rhetorical. Mancini's biographical and critical parts of the monograph constitute a valuable contribution to the history of early Italian mathematics.

FLORIAN CAJORI.

COLORADO COLLEGE,
COLORADO SPRINGS, COLO.

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[1] See P. Duhem, *Études sur Léonard de Vinci*, Paris, 1906.

[2] Vasari, *Vita di Pier della Francesca*, Firenze, 1550, 1568. Egnatio Danti, *Commentari alle due Regole di prospettiva di I. Barozzi*, about 1575.

[3] *Elogio di fra Luca Pacioli, Scritti inediti del P. D. Cossali, pubblicati da B. Boncompagni.* Roma, 1857. H. Staigmüller, *Lucas Paciuolo, eine biographische Skizze* (Zeitschr. f. Math. u, Phys., Bd. XXXIV, 1889).

[4] G. Pittarelli in *Atti del IV. Congresso dei matematici*, tom. III, Roma, 1909.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

471. Proposed by E. T. BELL, University of Washington.

If there is an infinite number of positive integers r for which the equation $\sum_{i=1}^{n} a_i^r = \sum_{i=1}^{m} b_i^r$ holds, where the a_i and b_i are given positive integers, prove that $m = n$, and that in some order the a_i are identical with the b_i.

472. Proposed by E. T. BELL, University of Washington.

If a_i and b_j ($i = 1, \cdots, n$; $j = 1, \cdots, m$) denote positive integers, and if $\sum_{i=1}^{n} a_i^r = \sum_{j=1}^{m} b_j^r$ for all odd positive integral values of r, prove that $m = n$, and that in some order the a_i are identical with the b_j.

GEOMETRY.

503. Proposed by J. W. CLAWSON, Ursinus College, Penn.

If two points A and B invert with respect to a third point O as center of inversion into A' and B', the middle point of the segment AB inverts into the point other than where the circle of Apollonius (the locus of a point P moving so that $A'P/PB' = A'O/OB'$) cuts the circle $OA'B'$.

504. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a given line. Find the envelop of the side of the pedal triangle opposite the moving vertex.

CALCULUS.

419. Proposed by C. C. YEN, Tangshan, North China.

Find the entire area of the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

420. Proposed by W. J. GREENSTREET, Stroud, England.

The join of the center of curvature of a curve to the origin is at an angle α to the initial line. Prove that with the usual notation,

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2 + \left(\frac{d^2p}{d\psi^2}\right)^2\right] = \frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

MECHANICS.

336. Proposed by C. N. SCHMALL, New York City.

An inclined plane, length l, makes an angle $\phi(< \pi/4)$ with the horizontal plane through its foot. From its foot, a body is projected upward along the plane, with a velocity equal to that of a falling body at the height h, so as to pass over the top and strike the horizontal plane at the *maximum distance* from the foot of the inclined plane. Show by the methods of the calculus that $x = h/\sin\phi\cos\phi$, and that the corresponding value of l is $2h\cot 2\phi/\cos\phi$.

337. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Assuming that a train may be accelerated by the application of a force equal to 1/40 of its gross weight and be braked with a force equal to 1/10 of its gross weight, show that the least time in which it may run from one to another of two stopping stations 5,000 feet apart is 2 minutes and 5 seconds. Also find the greatest speed during the run to be 54-6/11 miles per hour.

NUMBER THEORY.

255. Proposed by FRANK IRWIN, University of California.

Given any arithmetical progression whose first term a and common difference d are relatively prime integers, and any finite set of positive integers $m_1, m_2, \cdots$ also relatively prime to d, it is required to determine an integer n such that the multiples of $m_1, m_2, \cdots$ may occupy the same positions in the series of natural numbers beginning with n as they do in the arithmetical progression. This is to say that if the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the progression are divisible by m_1, so also will be the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the series $n, n + 1, n + 2, \cdots$, etc. Show that n may be determined as the solution of a congruence $An + B \equiv 0 \pmod{C}$, whose coefficients, A, B, are constants independent of the number and value of the m's.

256. Proposed by FRANK IRWIN, University of California.

Let p be an odd prime, and let the notation $1/k$ stand for the solution of $kx \equiv 1 \pmod{p}$. Then show that if the sum of the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{(p-1)/2}$$

be congruent to zero (mod p)—should that be possible—the same is true for the sum of their products two at a time, as well as four at a time.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A solution of 450 was received from G. Y. SOSNOW and one of 454 from J. J. GINSBURG which have not been acknowledged.

458. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that n terms of the series $1 + 3 + 4 + 6 + 7 + 9 + 10 + \cdots$ is $\frac{1}{4}(n + 1)(3n - 1)$ when n is odd, and $\frac{n}{2}\left(\frac{3n}{2} + 1\right)$ when n is even.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

This series is a combination of two arithmetical progressions $3 + 6 + 9 + \cdots$ and $1 + 4 + 7 + 10 + \cdots$.

When n is even, the first becomes

$$3 + 6 + 9 + \cdots + \frac{3}{2}n = \frac{n}{4}\left(\frac{3}{2}n + 3\right)$$

and the second,

$$1 + 4 + 7 + 10 + \cdots + \left(\frac{3}{2}n - 2\right) = \frac{n}{4}\left(\frac{3}{2}n - 1\right).$$

The sum of these two is

$$\frac{n}{4}\left(\frac{3}{2}n + 3\right) + \frac{n}{4}\left(\frac{3}{2}n - 1\right) = \frac{n}{2}\left(\frac{3}{2}n + 1\right).$$

When n is odd, the first becomes

$$3 + 6 + 9 + \cdots + \frac{3}{2}(n - 1) = \frac{n - 1}{4}\left(\frac{3n}{2} + \frac{3}{2}\right)$$

and the second,

$$1 + 4 + 7 + 10 + \cdots + \left(\frac{3n}{2} - \frac{1}{2}\right) = \frac{n + 1}{4}\left(\frac{3n}{2} + \frac{1}{2}\right).$$

The sum of these two is

$$\frac{n - 1}{4}\left(\frac{3(n + 1)}{2}\right) + \frac{n + 1}{4}\left(\frac{3n + 1}{2}\right) = \frac{1}{4}\left(\frac{3n^2}{2} - \frac{3}{2} + \frac{3n^2}{2} + 2n + \frac{1}{2}\right)$$

$$= \frac{1}{4}(3n^2 + 2n - 1) = \frac{1}{4}(n + 1)(3n - 1).$$

Also solved by L. A. H. WARREN, W. C. EELLS, GEORGE PAASWELL, E. H. WORTHINGTON, HORACE OLSON, G. L. WAGAR, H. N. CARLETON, E. J. OGLESBY, H. C. FEEMSTER, BENJ. SINITSSCY, W. J. THOME, C. C. YEN, R. M. MATHEWS, E. F. CANADY, N. P. PANDYA, J. J. GINSBURG, O. S. ADAMS, A. M. HARDING, and PAUL CAPRON.

GEOMETRY.

485. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the surface generated by the orthogonal projection of a given line upon a variable plane turning about a fixed axis.

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

Take the fixed axis as X-axis, and the common perpendicular of this line and the given line as Z-axis. Call the angle θ made by the line we are projecting and the X-axis, so that the given line has the direction cosines $\cos\theta$, $\sin\theta$, 0; let the coördinates of the point where this line intersects the Z-axis be $(0, 0, a)$. Then the equation of the variable plane may be written in the form

(A) $y - \lambda z = 0$, where λ is a parameter.

The equation of any plane through the given line may be written in the form

(B) $\sin\theta x - \cos\theta y + k(z - a) = 0$, where k is an arbitrary constant.

If (B) is perpendicular to (A), k must equal $-\cos\theta/\lambda$. Substituting this value for k in (B), (A) and (B) give the required equation in parameter form. Eliminating λ, we obtain the explicit equation

$$\tan\theta \cdot xy = y^2 + z^2 - az.$$

Rotating the axes through an angle $\theta/2$ about the Z-axis, and changing the origin to the point $(0, 0, a/2)$, the equation finally becomes

$$x^2\left(\frac{1-\cos\theta}{2\cos\theta}\right) - y^2\left(\frac{1+\cos\theta}{2\cos\theta}\right) - z^2 = -\frac{a^2}{4},$$

which is the equation of a hyperboloid of one sheet.

II. SOLUTION BY THE PROPOSER.

The orthogonal projection u of a given line p upon a plane β passing through a given axis q, may be obtained as the intersection of β with the plane α perpendicular to β and passing through p. When the plane β turns about q, the projecting plane α turns about p; to each position of β corresponds one and only one position of α, and vice versa. The line u is, therefore, the intersection of two corresponding planes α, β of two projective pencils (p), (q).

I. The lines p, q are coplanar. *The line u generates a cone of second degree (C), of which p, q are elements.*

The tangent planes to (C) along p and q are the planes perpendicular to the plane (pq) and passing through the lines p and q respectively. The line of intersection of these tangent planes is perpendicular to (pq) and is the polar ray of this plane with respect to (C). Hence: *The plane of the two given lines is a plane of symmetry of the cone.*

The orthogonal projection P' of any point P of p upon the plane β lies on the line u and in the plane π through P perpendicular to q. When β varies, P' describes, in the plane π, a circle having for diameter the segment joining P to the point of intersection of π with q. This circle is the curve of intersection of (C) with the plane π. Similarly for the orthogonal projection Q' of any point Q of q upon α. Hence: *The planes perpendicular to the given lines are the planes of the circular sections of the cone.*

This cone is sometimes called the *Orthogonal* cone (Theodor Reye, *Geometrie der Lage*, part I, p. 119, fifth edition).

If the given lines p, q are parallel, the cone becomes a cylinder of revolution.

II. The lines p, q are skew. *The line u generates a hyperboloid of one sheet (H), of which p, q are two rays of the same system.*

The planes perpendicular to the given lines are the planes of circular sections of the hyperboloid. Same proof as for the cone above.

If the two pencils of planes (p), (q) be transported parallel to themselves so as to have their axes p, q pass through the center of (H), they will generate the asymptotic cone of the surface. Hence: *The asymptotic cone of the hyperboloid is orthogonal.*

This hyperboloid is sometimes called the *Orthogonal* hyperboloid (Salmon, *Analytic Geometry of Three Dimensions*, Vol. I, p. 116, fifth edition).

REMARK.—From the method of construction of the line u it follows that the cone (C) (or the hyperboloid) is also the surface generated by the orthogonal projection of the line q upon a variable plane turning about the axis p.

The reader may consider the special cases when (1) the given lines are perpendicular to each other; (2) one or both of the given lines are at infinity.

Also solved by H. C. FEEMSTER, G. W. HARTWELL, and FRANK H. LOUD.

486. Proposed by ARON INGVALE, Brooklyn, N. Y.

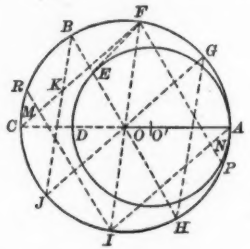
Does the following construction trisect an angle? With the vertex, O, of the given angle as center and with a radius R, describe a circle intersecting the sides of the given angle in A and B. With a radius $\frac{3}{4}R$, and center O' on OA, describe a circle tangent to the other circle at A and cutting the other side of the angle at E. At E draw a tangent to the last circle and produce it to meet the first circle at F. Draw FO. Then is angle BOF one-third of the angle BOA?

REMARK. Though the construction does not, of course, lead to the trisection of an angle in general, yet as a first approximation it is very good. This fact together with the fact that the construction is very simple, and that the proposer's demonstration that it does trisect the angle is very illusive, are the reasons for giving the problem a place in the MONTHLY.

EDITORS.

SOLUTION BY THE PROPOSER.

Let OG bisect $\angle AOF$. Extend FO and GO till they meet the original circle at I and J respectively. Also extend BO to H. Draw FC and AI, which for obvious reasons are parallel to each other and to GJ.



Then draw FP and IR each parallel to BH and prove $\angle BJG = \angle FIA$ as follows:

$\triangle BOJ$ is isosceles and since $CF \parallel JG$, therefore $\triangle KEB$ is also isosceles. But $\triangle KEB$ is similar to $\triangle FNI$ because their sides are respectively parallel.

Hence $\triangle FNI$ is isosceles and also $\triangle IMF$ to which it is congruent.

Hence $\angle CFI = \angle IFP = \angle BJG = \angle FIA$, from which follows $\angle BOF = \angle GOF$. But since $\angle GOF$ was constructed equal to $\angle AOC$, it follows that $\angle AOB$ is trisected.

Note.—The fallacy in this proof has been pointed out to Mr. Ingvale and he admits his error, but it is left as an exercise to others to show that the method applies to a right angle or a straight angle but fails in general. EDITORS.

CALCULUS.

403. Proposed by C. N. SCHMALL, New York City.

A paraboloid of revolution generated by the curve $x^2 = 4ay$, contains a quantity of water such that if a sphere of radius r be dropped to the bottom, it will just be covered by the water. Show that if the volume of water used in this experiment is to be a *minimum*, then we must have $a = r/6$.

Solution by H. S. Uhler, Yale University.

Since the surfaces are figures of revolution we may confine our attention to a parabola with its axis vertical and a co-planar circle whose center lies in this axis. Let the coördinates of the center of the circle be $(0, h)$. The equation of the circle is

$$x^2 + (y - h)^2 = r^2 .$$

Elimination of x between this equation and $x^2 = 4ay$ gives

$$y^2 - 2(h - 2a)y - (r^2 - h^2) = 0.$$

In order that the sphere may touch the paraboloid the last equation must have equal roots, the necessary and sufficient condition being

$$4(h - 2a)^2 + 4(r^2 - h^2) = 0, \quad \text{or} \quad h = (r^2 + 4a^2)/4a.$$

The volume of the paraboloid up to the level $h + r$ is given by

$$\pi \int_0^{h+r} x^2 dy = 4\pi a \int_0^{h+r} y dy = 2\pi a(h + r)^2 = \frac{\pi(r + 2a)^4}{8a}.$$

In general, therefore, the volume of the water is expressed by

$$v \equiv \frac{\pi(r + 2a)^4}{8a} - \frac{4\pi r^3}{3}.$$

Two cases now present themselves, namely, (i) a is kept constant while r varies, or (ii) r is fixed in value while a changes.

Case (i)

$$\frac{\partial v}{\partial r} = \frac{\pi[r(r - a)^2 + 11a^2 r + 8a^3]}{2a}.$$

Since this expression cannot vanish for any positive value of r the necessary condition for maxima and minima is not fulfilled, as is also apparent from general considerations.

Case (ii)

$$\frac{\partial v}{\partial a} = \frac{\pi(r + 2a)^3(6a - r)}{8a^2},$$

and

$$\frac{\partial^2 v}{\partial a^2} = \frac{\pi}{4a^3}[3a(r + 2a)^3 + (6a - r)(a - r)(r + 2a)^2].$$

When $6a - r = 0$, the first and second derivatives become 0 and $+ 64\pi r$, respectively. Consequently, when $a = r/6$ a minimum of volume is attained. It should be remarked, however, that $h = 5r/3$ and $h - r = 2r/3$, so that the lowest point of the sphere is two-thirds of the radius above the vertex of the paraboloid; in other words, the sphere has not literally been "dropped to the bottom."

Also solved by A. W. Smith, F. A. Pottle, Horace Olson, J. W. Clawson, C. A. Nickle, O. S. Adams, Paul Capron, J. A. Bullard, Geo. W. Hartwell, J. A. Caparo, A. G. Rau, H. C. Feemster, and G. Paaswell.

404. Proposed by B. J. Brown, Victor, Colorado.

Solve the differential equation, $(x^2 - y^2)(1 + dy/dx) = 2xy(1 - dy/dx)$.

Solution by H. L. Agard, Williams College.

The equation may be written,

$$(x^2 - 2xy)dx + (2xy - y^2)dy + x^2 dy - y^2 dx = 0,$$

which, after adding and subtracting $x^2 dx$ and $y^2 dy$, becomes,

$$2x(x - y)dx + 2y(x - y)dy - (x^2 + y^2)(dx - dy) = 0.$$

Introducing the integrating factor $(x-y)^{-2}$, we have,

$$\frac{(x-y)(2xdx+2ydy)-(x^2+y^2)(dx-dy)}{(x-y)^2}=0,$$

which, upon integration, becomes

$$\frac{x^2+y^2}{x-y}=c, \quad \text{or} \quad x^2+y^2-c(x-y)=0.$$

Also solved by A. W. SMITH, NORMAN ANNING, J. W. CLAWSON, J. A. BULLARD, G. PAASWELL, O. S. ADAMS, ELIJAH SWIFT, FREDERICK WOOD, HORACE OLSON, C. A. BARNHART, L. M. COFFIN, G. W. HARTWELL, J. D. BOND, A. G. RAU, C. A. HUTCHINSON, CLARIBEL KENDALL, C. S. ATCHINSON, J. W. CROMWELL, C. P. SOUSLEY, J. A. CAPARO, and PAUL CAPRON.

405. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the greatest quadrilateral which can be formed with the four given sides a, b, c, and d taken in order.

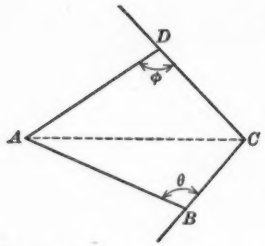
SOLUTION BY A. M. HARDING, University of Arkansas.

In the quadrilateral $ABCD$, if $AB=a$, $BC=b$, $CD=c$, $AD=d$, we have

$$\overline{AC}^2=a^2+b^2-2ab\cos\theta=c^2+d^2-2cd\cos\phi.$$

Hence,

$$ab\cos\theta-cd\cos\phi=\frac{a^2+b^2-c^2-d^2}{2}. \tag{1}$$



Also, area $=\frac{1}{2}ab\sin\theta+\frac{1}{2}cd\sin\varphi$. For a maximum or minimum we must have

$$ab\cos\theta d\theta+cd\cos\varphi d\varphi=0. \tag{2}$$

From (1) we obtain, by differentiation,

$$-ab\sin\theta d\theta+cd\sin\varphi d\varphi=0. \tag{3}$$

It follows from (2) and (3) that

$$\tan\phi=-\tan\theta; \quad i.e., \quad \phi+\theta=180^\circ. \tag{4}$$

Therefore,

$$\cos\theta=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

Hence, the given quadrilateral will be a maximum when it can be inscribed in a circle.

Note.—It is evident from the nature of the problem that it is necessary to consider only *convex* quadrilaterals, and that (4) gives a maximum and not a minimum.

406. Proposed by C. N. SCHMALL, New York City.

Given $f(x+h)+f(x-h)=f(x)\cdot f(h)$, determine by Taylor's theorem or otherwise the nature of the function f.

SOLUTION BY W. M. CARRUTH, Hamilton College, New York.

Given

$$f(x+h)+f(x-h)=f(x)\cdot f(h). \tag{1}$$

Put $h=0$ in this equation. Then $2\cdot f(x)=f(0)\cdot f(x)$. Hence, either

$$f(x)=0, \tag{2}$$

which is a trivial solution of the problem and will be ignored until the end of the discussion, or

$$f(0) = 2. \tag{3}$$

Changing the sign of h in (1) does not alter the left-hand member; hence,

$$f(x)\cdot f(h) = f(x)\cdot f(-h),$$

or $f(h) = f(-h)$. Differentiating this, $f'(h) = -f'(-h)$; whence

$$f'(0) = 0. \tag{4}$$

Differentiate (1) twice with respect to h. Then, $f''(x+h) + f''(x-h) = f(x)\cdot f''(h)$. Putting $h = 0$ in this equation, we get

$$2\cdot f''(x) = f''(0)\cdot f(x). \tag{5}$$

Replace $f''(0)$, which is a constant, by $\pm 2c^2$; and put y for $f(x)$, and d^2y/dx^2 for $f''(x)$. Although the prime was used to represent differentiation with respect to h, (5) is an identity and the change here to x is legitimate. (5) may now be written

$$\frac{d^2y}{dx^2} \mp c^2y = 0. \tag{6}$$

Using the upper sign in (6), the solution may be written in the form $y = A\cdot\cosh(cx + B)$, where A and B are constants to be determined. From (3) and (4) above,

$$A\cdot\cosh B = 2, \quad \text{and} \quad Ac\cdot\sinh B = 0.$$

Therefore, if $c \neq 0$, we have $B = 0$ and $A = 2$ (the values $B = n\pi i$, $A = (-1)^n\cdot 2$ are no more general than these). Hence, one solution of (1), which is easily verified, is

$$f(x) = 2\cdot\cosh cx. \tag{7}$$

(The fact that (7), and (8) below, may be verified on substitution in (1) proves that the constant c is an arbitrary constant.)

Using the lower sign in (6), the solution may be written in the form

$$y = A\cdot\cos(cx + B).$$

As before,

$$A\cdot\cos B = 2, \quad \text{and} \quad Ac\cdot\sin B = 0;$$

whence

$$B = 0, \quad \text{and} \quad A = 2.$$

Another solution of (1) is, therefore,

$$f(x) = 2\cdot\cos cx. \tag{8}$$

If $c = 0$ in (6), we have the solution $y = Ax + B$. Here, from (3) and (4) as before, $A = 0$, and $B = 2$. But the solution $f(x) = 2$ is included in both (7) and (8), if $c = 0$ is permitted there.

Collecting the results that have been given in (2), (7), and (8), we may say that the function $f(x)$ defined by equation (1) must be equivalent to one of the forms

$$0, \quad 2\cdot\cosh cx, \quad 2\cdot\cos cx,$$

where c is any real number. If imaginary values of c are permitted, the last two forms of $f(x)$ are equivalent.

If a solution by Maclaurin's theorem rather than by differential equations is desired, the work following equation (5) above may take this form:

Replace $f''(0)$, which is a constant, by $\pm 2c^2$, and consider the primes to mean differentiation with respect to x. Then (5) may be written

$$f''(x) = \pm c^2 f(x).$$

Successive differentiation of this equation leads to the general result

$$f^{(n+2)}(x) = \pm c^2 f^{(n)}(x),$$

or, if $x = 0$ be substituted,

$$f^{(n+2)}(0) = \pm c^2 f^{(n)}(0). \tag{6'}$$

From (4) and (6'), all the odd ordered derivatives of $f(x)$ are zero when $x = 0$. By successive applications of (6') we have (since $f''(0) = \pm 2c^2$)

$$f^{IV}(0) = 2c^4, \quad f^{VI}(0) = \pm 2c^6, \cdots, f^{(4n)}(0) = 2c^{4n}, \quad f^{(4n+2)}(0) = \pm 2c^{4n+2}.$$

Hence, by Maclaurin's theorem,

$$f(x) = 2 \pm 2c^2x^2/2! + 2c^4x^4/4! \pm 2c^6x^6/6! + \cdots.$$

This gives either

$$f(x) = 2 \cdot \cosh cx, \quad \text{or} \quad f(x) = 2 \cdot \cos cx,$$

to which solutions $f(x) = 0$ should be added as before.

Also solved by H. C. FEEMSTER, OSCAR S. ADAMS, PAUL CAPRON, and the PROPOSER.

MECHANICS.

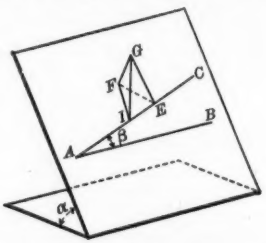
319. Proposed by LAENAS G. WELD, Pullman, Illinois.

A hexagonal pencil lies upon the inclined top of a drawing table and is on the point of either rolling or sliding. Find the angle between its direction and the horizontal edge of the table, the coefficient of friction being μ.

SOLUTION BY H. S. UHLER, Yale University.

The condition for being on the verge of sliding is expressed by $\mu = \tan \alpha$, where α denotes the angle between the table top and the horizontal and is sometimes called the "limiting angle of repose." The proof of this relation seems superfluous in this place because it is given in practically all text-books which are devoted in part or entirely to elementary mechanics.

In order that the pencil may be on the point of rolling, it is obviously necessary and sufficient that the vertical line through the center of gravity intersect the lateral edge which is at the lowest level. In the diagram let $\overline{AB}$ and $\overline{AC}$ indicate respectively a horizontal line on the table top and the lowest lateral edge of the hexagonal prism. G marks the center of gravity which is assumed to lie in the geometric axis of the pencil. $\overline{GI}$ denotes the vertical through G, that is,



the line of action of the weight of the pencil which intersects the edge $\overline{AC}$ in the point I. $\overline{GE}$ and $\overline{GF}$ are perpendiculars dropped from G upon the edge $\overline{AC}$ and the table top, respectively. In other words, the plane EFG contains a right section of the prism passing through the center of gravity. Without quoting the well-known theorems of elementary geometry we see at once that $\angle EFI = \beta$, $\angle EGF = 30°$, $\angle FGI = \alpha$, $\angle EFG = \angle FEI = \angle GFI = 90°$. Consequently, $\overline{FG}/\overline{EF} = \cot 30° = \sqrt{3}$, $\overline{EF}/\overline{FI} = \cos \beta$, and $\overline{FI}/\overline{FG} = \tan \alpha$. Multiplying these three equations together we obtain $1 = \sqrt{3} \cos \beta \tan \alpha$, but $\tan \alpha = \mu$, hence

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}\mu}\right).$$

Remark.—Since the greatest value of a cosine is unity the least value of μ is equal to $1/\sqrt{3} \doteq 0.57735$ corresponding to $\alpha = 30°$ and $\beta = 0°$, as it should be.

Also solved by W. J. THOME, M. R. BOWERMAN, and GEORGE PAASWELL.

320. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A heavy uniform chain of length l is hung over a rough horizontal cylinder of radius r. Show that one end of the chain will be $[2\mu r/(\mu^2+1)](e^{\mu\pi}+1)+l(e^{\pi\mu}-1)$ units lower than the other, just when the chain begins to move, the coefficient of friction being μ.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let x = the longest part hanging down, and $l-(\pi r+x)$ = the shortest, θ being the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right; and T, T' the tensions at any points in those parts of the string to which x and $l-(\pi r+x)$ belong. Then if C be a constant, we have

$$T = Ce^{\mu\theta} + \frac{r}{\mu^2+1}\{2\mu\sin\theta + (1-\mu^2)\cos\theta\}, \tag{1}$$

by theory, μ and θ being both negative, the string being supposed to move in the direction of x.

When $\theta = -\pi/2$, $\mu = -\mu$, and $T = x$, a unit of length of string weighing a unit, we have

$$C = \left(x - \frac{2\mu r}{\mu^2+1}\right)e^{-(1/2)\pi\mu}, \tag{2}$$

and (1) then becomes

$$T = \left(x - \frac{2\mu r}{\mu^2+1}\right)e^{[\theta-(\pi/2)]\mu} + \frac{r}{\mu^2+1}\{-2\mu\sin\theta + (1-\mu^2)\cos\theta\}. \tag{3}$$

When $\theta = 0$,

$$T = \left(x - \frac{2\mu r}{\mu^2+1}\right)e^{-(1/2)\pi\mu} + \frac{r(1-\mu^2)}{\mu^2+1}, \tag{4}$$

which is the tension at the vertex.

Again, $T' = l-(\pi r+x)$, when $\theta = \pi/2$, $\mu = -\mu$; then

$$Ce^{-(1/2)\pi\mu} - \frac{2\mu r}{\mu^2+1} = l-(\pi r+x),$$

or

$$C = \left\{l-(\pi r+x) + \frac{2\mu r}{\mu^2+1}\right\}e^{(1/2)\pi\mu}. \tag{5}$$

Hence

$$T' = \left\{l-\pi r-x+\frac{2\mu r}{\mu^2+1}\right\}e^{\mu[\theta+(\pi/2)]} + \frac{r}{\mu^2+1}\{-2\mu\sin\theta+(1-\mu^2)\cos\theta\}. \tag{6}$$

When $\theta = 0$,

$$T' = \left\{l-\pi r-x+\frac{2\mu r}{\mu^2+1}\right\}e^{(1/2)\pi\mu} + \frac{r(1-\mu^2)}{\mu^2+1}. \tag{7}$$

For equilibrium, $T = T'$, and then

$$\left(l-\pi r-x+\frac{2\mu r}{\mu^2+1}\right)e^{(1/2)\pi\mu} = \left(x-\frac{2\mu r}{\mu^2+1}\right)e^{-(1/2)\pi\mu}, \tag{8}$$

giving

$$x = \frac{l-\pi r}{1+e^{-\pi\mu}} + \frac{2\mu r}{\mu^2+1}, \tag{9}$$

and

$$l-(\pi r+x) = \frac{(l-\pi r)e^{-\pi\mu}}{1+e^{-\pi\mu}} - \frac{2\mu r}{\mu^2+1}. \tag{10}$$

(9) and (10) give the two vertical parts of the string, their difference being

$$\frac{(e^{\pi\mu}-1)(l+\pi r)}{1+e^{-\pi\mu}} + \frac{4\mu r}{\mu^2+1},$$

not agreeing with the required difference in the statement of the problem.

Also solved with same result by PAUL CAPRON, RALPH E. ROOT, and ELIJAH SWIFT.

NUMBER THEORY.

217. (May, 1914.) Proposed by E. T. BELL, University of Washington.

(i) If r is a prime greater than 2, and $p = 2^a r + 1$ is prime, the only solution, when n is greater than 2, of $x^n - y^n = p$, is $n = 3$, $x = 2$, $y = 1$.

(ii) The only primes that are simultaneously of the forms $4k + 1$ and $3^m - 2^m$ are 1 and 5.

(iii) Generalize (ii).

SOLUTION BY FRANK IRWIN, University of California.

(i) Since $x^n - y^n$ is divisible by $x - y$, we must, if it is to be a prime, have $x - y = 1$, or $x = y + 1$.

Again, n must be a prime; for $x^{st} - y^{st}$ is divisible by $x^s - y^s$, and cannot, therefore, be a prime.

Now $2^a r = p - 1 = x^n - y^n - 1 = (y+1)^n - y^n - 1 = ny^{n-1} + \binom{n}{2} y^{n-2} + \cdots + ny$; so that $2^a r$ is divisible by ny, since n, being a prime, divides each of the binomial coefficients. Consequently we must have $n = r$ (n is greater than 2 by hypothesis) and y a power of 2.

But by writing $2^a r$ as $x^n - (x-1)^n - 1$, we may show, just as above, that x also is a power of 2.

Now as $x = y + 1$, these results are consistent with each other only if $y = 1$, $x = 2$.

In this last case we have $2^a r + 1 = 2^n - 1^n = 2^r - 1$, since $n = r$. Hence, $2^a(2^{r-a} - r) = 2$, and, therefore, $a = 1$ and $2^{r-a} - r = 1$, that is, $2^{r-1} = r + 1$; so that $r = n = 3$.

(ii) $3^m - 2^m = (2+1)^m - 2^m = m2^{m-1} + \binom{m}{2} 2^{m-2} + \cdots + m2 + 1$, and can be of the form $4k + 1$ only if m is even (unless $m = 1$, when $3^m - 2^m = 1$).

But if m is even, $= 2n$, say, $3^m - 2^m = 9^n - 4^n$ and is therefore divisible by $9 - 4$ or 5, the other factor being greater than 1, unless $n = 1$, and is, therefore, prime in this last case only.

(iii) A parallel argument leads to the result that the only primes that are simultaneously of the forms $a^2k + 1$ and $(a+1)^m - a^m (a > 1)$ are 1 and, possibly, $(a+1)^a - a^a$. This latter number may or may not be prime as the cases $a = 3, 4$ show.

Also solved by MARY E. CARTER.

221. (September, 1914.) Proposed by T. E. MASON, Bloomington, Indiana.

Find a number x such that the sum of the divisors of x is a perfect square. [CARMICHAEL, *Theory of Numbers*, p. 17.]

SOLUTION BY E. B. ESCOTT, Kansas City, Mo.

(1) Let x be prime. Then $x + 1 = r^2$, or $x = (r+1)(r-1)$. This is possible only if $r = 2$; otherwise, x is not prime. Hence, $x = 3$ satisfies the condition.

(2) Let $x = abc\cdots$, where $a, b, c, \cdots$ are different primes. Denoting the sum of the divisors of x by Σx,

$$\Sigma x = (a+1)(b+1)(c+1)\cdots.$$

Forming a table of factors of $a + 1$, it is very easy to pick out products of two or more which shall be square, *e. g.*,

x	Σx	x	Σx
2.11	6^2	7.97	28^2
2.47	12^2	.	.
11.47	24^2	.	.
5.23	12^2	.	.
5.53	18^2	2. 5.7	12^2
23.53	36^2	5. 7.11	24^2
7.17	12^2	5. 7.47	48^2
7.31	16^2	2.13.41	42^2
7.71	24^2	5.67.101	204^2
		.	.
		.	.
		.	.

(3) Corresponding to a factor a^2 in the number, we have in the sum of the divisors $1 + a + a^2$, etc. By forming a table of $1 + a + a^2$, $1 + a + a^2 + a^3$, $\cdots$ we can find examples where the number contains square factors, cube factors, $\cdots$.

EXAMPLES:

x	Σx	x	Σx
$7^2.11^2.5.13$	798^2	$3^3.7.19$	80^2
$41^3.83$	2436^2	$2^2.5.41$	42^2
$41^3.5.13$	2436^2	.	.
$3^3.89$	60^2	.	.
		.	.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kans.

REPLIES.

31. What are the actual courses now offered in colleges and universities in this country for the preparation of teachers (1) for secondary schools, (2) for colleges? The discussion may well lead to the consideration also of what courses *should be offered* for the preparation of teachers of mathematics (1) for secondary schools, and (2) for colleges.

REPLY BY U. G. MITCHELL, University of Kansas.

In the American Report, Committee No. V, of the International Commission on the Teaching of Mathematics, published in 1911, we find (pp. 5–6) these statements:[1]

"Twenty years ago no professional training of university grade existed in this country to prepare teachers of mathematics for secondary schools. At that time the young teacher's sole preparation for his work was the taking of as many academic courses as possible, plus, in some instances, a course on the history of education or some lectures on pedagogy. . . . About 15 years ago we find conditions throughout the country beginning to change in this respect. At least five different educational institutions had by this time (1895) established courses on the teaching of algebra and geometry, which, together with a course on general pedagogy, formed a certain professional training for high school teaching in mathematics. Up to 1900 only four other colleges are known to this committee to have added courses in the pedagogy of secondary mathematics to their programs.

"The past 10 years have shown far greater interest in pedagogical matters and a much more rapid growth in courses of this kind. At present (1910) about 25 other colleges in addition to those above mentioned, have developed such courses."

The writer has just completed an examination of the most recent catalogs from a selected list of 100 colleges and universities to see how many of them are now offering courses especially designed for students preparing to teach mathematics. The result shows that the change referred to above is continuing. Fifty-three of the 100 institutions are offering one or more courses of the following four different classes:

(1) *Courses especially designed for teachers of secondary mathematics.* Such courses were listed by 40 of the 100 institutions and were variously designated as "Teachers' Course," "Mathematical Methods," "History and Teaching of Mathematics," "Teachers' Course in Algebra and Geometry," etc. In five cases "reviews of secondary mathematics with special attention to methods"

[1] *Bulletin, 1911, No. 12, U. S. Bureau of Education* (Washington, D. C.).

constituted a part or all of the course. In a considerable number of cases it was stated that courses in analytic geometry and calculus were prerequisites. The credit given for such courses varied from 1 to 6 hours.

(2) *Courses in practice teaching.* In 6 of the 100 institutions students were offered opportunity to do actual teaching under the direction of supervisors in training schools.

(3) *Courses in the history of mathematics.* Such courses were offered in 24 of the 100 institutions and, with a single exception, were given either two or three hours credit. Two institutions offered more than one such course. Nine of the institutions which gave no separate course in history of mathematics stated that a part of the time of the teachers' course was devoted to history. Hence, some formal teaching in the history of mathematics was offered in at least 33 of the 100 institutions.

(4) *Courses in the foundations of mathematics.* Such courses were offered in 8 of the 100 institutions. They were variously designated as "Fundamental Concepts of Mathematics," "Fundamental Theorems of Algebra and Geometry," "Foundations of Geometry," etc., and were generally three-hour courses.

The writer failed to find any courses which were said to be designed especially to prepare teachers of collegiate mathematics. It is quite possible, however, that some of the history and foundation courses have been established primarily for students who expect to become professors of mathematics. In this respect, the condition is practically the same as described by the American Report, Committee X, of the International Commission on the Teaching of Mathematics in the following language[1] (italics mine):

> "There is a universal feeling that courses in the pedagogy of mathematics are of very small advantage to the future college teacher. Those professors who are willing to see such courses introduced specify either that they should be in addition to all the present courses in mathematics and not a substitute for any purely mathematical subject, or that they should be introduced only to satisfy the imperative demands of the schools.
>
> "A wiser view seems to be that which argues that the college teacher can well afford to spend some time in learning the best methods of teaching, whatever his subject; and that a course in the pedagogy of mathematics will be of very great value to the teacher of mathematics when the graduate school or university shall have developed a course suited to his needs. *There appears to be sufficient material at hand, and it is strange that no American institution has solved the problem of a course in mathematical pedagogy in such a manner as to appeal to the professor of mathematics.*"

As to what courses should be offered for the preparation of teachers of secondary mathematics, a considerable number of American institutions are now offering and, in large part, requiring, the following standard suggested as a minimum by Professor J. W. A. Young at the Cambridge meeting[2] (1912) of the International Commission:

(*a*) Trigonometry, college algebra, analytic geometry.
(*b*) Surveying, or descriptive geometry, or elementary astronomy.
(*c*) The differential and integral calculus with applications to geometry, mechanics and physics.
(*d*) Modern geometry.

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[1] *Bulletin, 1911, No. 7, U. S. Bureau of Education* (Washington, D. C.), pp. 23–4.
[2] See *L'Enseignement Mathématique* for Nov. 15, 1912, p. 483.

(*e*) The elements of analytic mechanics.
(*f*) The elements of theoretic and laboratory physics.
(*g*) Algebra from a modern standpoint.
(*h*) One or more courses introductory to important fields of modern mathematics.
(*i*) One or more courses on the history of mathematics.
(*j*) One or more courses on the teaching of mathematics.

At the university with which the writer is connected, a student majoring in mathematics is required to take before graduation at least 30 hours of work in pure mathematics courses. Besides the usual courses in college algebra, trigonometry, analytical geometry and calculus (15 hours), a two-hour course in higher algebra, a three-hour course in modern geometry, a two-hour course in history of mathematics and a three-hour course in analytic mechanics or advanced calculus, are specifically required. The remaining 5 or more hours are elective. In order to be certificated for teaching mathematics in the secondary schools in Kansas the student must also take a three-hour teachers' course in mathematics, a semester's work in practice teaching in the training school, a three-hour course in general psychology, a three-hour course in educational psychology, a three-hour course in the history of education and 6 hours of elective work in the school of education. While these requirements are probably less than some others, they doubtless do not differ greatly from those of a number of middle-western universities.

The writer believes that courses in education possess a distinct value for the prospective teacher of mathematics. They give him some conception of the human side of his work. More than one young professor, splendidly equipped mathematically, has failed as a teacher because he spent his energy in trying to teach mathematics *to* students and never acquired the point of view of teaching students *by means of* mathematics.

REPLY BY R. C. ARCHIBALD, Brown University.

Within the past decade many of those charged with directing the education of youth in the United States have had their outlook immensely extended, and their ideas radically changed by careful study and consideration of methods employed in other countries. They have discovered that the standards to which teachers in practically all the secondary schools of France, Germany, Italy and many other countries must attain are far higher than those demanded in even the best secondary schools of the United States. As far as mathematics is concerned this has been set forth with great explicitness, during the past few years through the publications inspired by the International Commission on the Teaching of Mathematics.

As a result, thoughtful inquirers have reached the conclusion that the efficiency and general well-being of this country demand that some radical changes be made in the method of conducting secondary education; moreover, that as a beginning of necessary reform, newly appointed teachers should be college graduates who are informed as to the problems and methods in secondary education and who, while they have learned to appreciate varied forms of scholarship and culture, have yet been specially equipped to teach one or two nearly related subjects.

It is with deep appreciation of the important work to be done in promoting the cause of secondary education, and therefore of higher education, that the department of mathematics in Brown University lays emphasis, by virtue of special courses and personal effort, on the methods employed in the preparation of teachers of mathematics. The annual increase in the number of students taking this work in the department seems to testify to the fact that a real need is being met.

It is considered important that prospective teachers of mathematics should have a thorough scientific reconsideration in college of the principles of secondary school mathematics, that the true inwardness of all operations, and a clear understanding of the foundations of the various subjects, should be acquired. This knowledge, together with adequate facility in presenting it, presupposes that the student has been carried far beyond the subject matter actually to be taught at a later day in the secondary school. But it is only with such preparation that a teacher of mathematics can claim to be competent to lay the foundations of a subject which is doubtless of more far-reaching importance than any other.

The eleven semester courses (each three hours per week) required for recommended teachers are: algebra, geometry, plane trigonometry, analytical geometry, and differential and integral calculus.

The two-semester *teachers' course in algebra* constitutes an introduction to some of the concepts of modern analysis. Among the topics treated are, the number system, with special reference to irrational numbers, limits, infinite series, fundamental operations, determinants and proofs of the fundamental theorem of algebra.

In the two-semester *teachers' course in elementary geometry* the student is taught: methods for attacking Euclidean problems; discussions of famous problems; the existence of transcendental numbers and the proof of the transcendence of π; means of rigorous discussion of the more delicate and difficult parts of the subject such as the systems of axioms; something about (*a*) the history and literature of elementary geometry and (*b*) the most important French, German and Italian texts.

As to the course in *solid and spherical geometry*, the ability to reason accurately from fundamental axioms, and to solve originals, is demanded.

The work in *analytical geometry* occupies one and one-half semesters and that in *calculus* two and one-half semesters.

In addition to those mentioned above it is possible to take six or eight other year-courses, partaking somewhat of a graduate character, and some such courses are usually elected by our embryonic secondary school teachers. For example, during the year 1915–16 Goursat's *Mathematical Analysis*, volume 1, was a text in one course; in another, Pierpont's *Functions of a Complex Variable*, and in a third, Papelier's *Coordonnées tangentielles*. In no one of these courses were there less than ten students, the majority of whom were preparing to be secondary school teachers. For the year 1916–17, the first volume of Weber's

Algebra is a text in one course and in another, in differential geometry, the texts are the last part of volume 1 of Goursat's *Mathematical Analysis*, Gauss's memoirs on *Curved Surfaces* and selected chapters from Humbert's *Cours d'Analyse*.

Another advantage in connection with our preparation of teachers is the meetings of the Mathematical Club. The club is a recent organization which aims to create an atmosphere in which to establish friendly intercourse among students and teachers, and to supply an opportunity for the presentation, by advanced students and professors, of papers of general interest.

Practically all students recommended as teachers of mathematics to secondary schools have had two or more courses in education besides work in science. Many students return for graduate work leading to the degree of Master of Arts. Their work is mainly in the department of education and in the schools of Providence and the surrounding cities. Under suitable restrictions tuition fees are met through scholarships awarded by the Board of Education of the state of Rhode Island. Such students (men and women) are not only well equipped scholastically, but have also acquired valuable insight and experience with reference to methods of imparting knowledge.

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

The distinguished mathematical physicist, Professor DON JOSÉ ECHEGARAY, of the University of Madrid, died on Sept. 15, aged eighty-three years.

Mr. F. S. NOWLAN, of Columbia University, has been appointed instructor in mathematics at the Carnegie School of Technology, Pittsburgh.

At Pennsylvania State College, Dr. J. E. ROWE has been promoted from assistant professor to associate professor of mathematics.

Mr. E. J. OGLESBY, instructor in mathematics in the University of Virginia, has been appointed professor of mathematics at the College of William and Mary.

Mr. ALBERT H. HOLMES, a contributor to the MONTHLY in the department of Problems and Solutions, and a charter member of the Association, died at Brunswick, Maine, Sept. 10, 1916, at the age of sixty-five years.

Dr. J. A. BULLARD, Mr. C. E. NORWOOD, and Mr. J. J. TANZOLA have been appointed instructors in mathematics at the U. S. Naval Academy, Annapolis, Maryland.

At Dartmouth College, Dr. R. D. BEETLE and Dr. F. M. MORGAN have been made assistant professors of mathematics, and Dr. C. H. FORSYTH, of the University of Michigan, has been appointed instructor in mathematics.

Dr. H. M. SHEFFER, of the College of the City of New York, has been appointed lecturer on philosophy at Harvard University, offering the courses in logic formerly given by the late Professor Royce.

Mr. J. W. LASLEY, JR., returns to the faculty of the University of North Carolina after a year's leave of absence as fellow by courtesy in mathematics at Johns Hopkins University.

The Mathematical Club of the University of North Carolina was organized on October 19th. The officers are: WILLIAM CAIN, president, ARCHIBALD HENDERSON, vice-president, and J. W. LASLEY, Jr., secretary-treasurer. The club proposes to engage in mathematical investigation, especially along historical and pedagogical lines.

Miss EDITH M. COON, who for the past two years has held a position as instructor in the department of mathematics at Mount Holyoke College, has gone to Madras, India, as vice-principal of the Christian College for Women that has been recently founded there under English and American auspices. Miss Coon will teach mathematics and physics.

The October number of *The Monist* is devoted to a commemoration of the scientific and philosophical work of GOTTFRIED WILHELM LEIBNITZ and its influence on modern thought. This material in celebration of the Leibnitz bicentenary has been collected and edited by Mr. P. E. B. JOURDAIN, the Cambridge scholar, who is especially interested in the fields of mathematics, physics, and logic.

A review of the Napier Tercentenary Memorial Volume by Professor L. C. Karpinski was published in *Science* for September 22, 1916. This volume contains thirty papers, touching a variety of interests related to logarithms, by representatives of the British Empire, France, Germany, Italy, Denmark, Turkey, and the United States. The papers by representatives of this country have already been noted in these columns in earlier issues.

At Sheffield Scientific School, Yale University, the following changes have been announced: Dr. H. F. MACNEISH, instructor in mathematics, has resigned to accept an appointment in the De Witt Clinton High School, New York City; Dr. D. D. LEIB has gone to the Connecticut College for Women, New London; Dr. P. R. RIDER has become instructor in mathematics in Washington University, St. Louis, Missouri; and Dr. D. F. BARROW, of the University of Texas, and Mr. J. K. WHITTEMORE have been appointed instructors in mathematics.

During the past year, the Mathematical Club of Rutgers College held regular monthly meetings at which the following papers were presented: "Tangencies of circles," Dr. J. A. Ingham; "The occurrence of conic sections in the

orbits of the heavenly bodies," Professor W. B. Stone; "Life insurance; the life tables," Professor E. Brasefield; "The use of the auxiliary angle," Professor Richard Morris; "Summation of certain trigonometric series," Mr. C. P. Osborne; "Logarithms; exposition of Napier's principles," Mr. Ed. S. Ingham; "Ptolemy's theorem and its relation to trigonometry," Mr. Edwin Florance; "The triangle and its circles," Mr. L. B. Gittleman.

It is with regret that we record the death of Dr. L. L. CONANT, professor of mathematics in the Worcester Polytechnic Institute, Worcester, Massachusetts. He was killed by an auto truck in front of his home on Oct. 11, 1916. Professor Conant was a graduate of Dartmouth College, class of 1879, A.M., 1887, and Ph.D. of Syracuse University, 1893. He engaged in public school work from 1879 to 1887, at which time he became professor of mathematics at the Dakota School of Mines, remaining in this position for three years. After one year's study at Clark University, he accepted the professorship of mathematics at Worcester in 1891, in which position he continued until his death. He was interested in all phases of education, serving not only in his official capacity at Worcester, but as a member of the Massachusetts State Board of Education from 1909–1914. Professor Conant was a member of the American Mathematical Society, the London Mathematical Society, the Mathematical Association of America, and other scientific societies. He was the author of "The Number Concept," "Exercises in Plane Geometry," and "Plane and Spherical Trigonometry with Tables." Professor Conant made a provisional bequest of $10,000 to the American Mathematical Society, the income to be offered once in five years as a prize for original work in pure mathematics.

The sixty-third annual meeting of the Indiana State Teachers' Association was held at Indianapolis on Oct. 26–28, attended by more than 10,000 teachers of the state representing all grades of school work from the kindergarten to the college. The general meetings were held in five sections, and the sectional meetings relating to special subjects and interests met in thirty different sections. The mathematics section composed of high school and college teachers of mathematics met during the forenoon of Oct. 26 in Cleb Mills Hall. Some three hundred teachers of mathematics were present. The following program was presented: "The hurdle system of teaching algebra," by Mr. M. A. DALMAN, of manual training high school, Indianapolis; "Efficiency tests in mathematics," by Professor D. A. ROTHROCK, of Indiana University; "Some school-room problems," by President R. J. ALEY, of the University of Maine; and a round table "Discussion of recent tendencies in opposition to mathematics in the secondary schools," participated in by Professor F. H. HODGE, of Franklin College, Professor E. N. JOHNSON, of Butler College, Professor R. B. STONE, of Purdue University, and by others. In this discussion Professor HODGE reviewed the criticisms of the high-school course in mathematics by Commissioner David Snedden, of Massachusetts, and Dr. ABRAHAM FLEXNER, of New York; Professor JOHNSON presented the sentiment of a representative group of "Who's

Who" men and women, showing some 90 per cent. of those expressing themselves as being strongly in favor of retaining mathematics in the curriculum of the high school.

The Mathematics Section of the Indiana State Teachers' Association dates back to 1891 at which time the College Association of Mathematics Teachers, which had been formed a few years earlier, took on the broader field of secondary as well as college mathematics. The programs are arranged so as to include topics of interest to mathematics teachers of any grade. At the close of the sectional meeting a call was made for teachers of college grade to meet and make preliminary organization with reference to forming an Indiana branch of the Mathematical Association of America. The preliminary organization was effected by the election of Professor S. C. DAVISSON, of Indiana University, chairman, Professor W. O. MENDENHALL, of Earlham College, secretary-treasurer, and Professor F. H. HODGE, of Franklin College, as member of the executive committee. This is the fifth section of the Association to be organized. The others are Kansas, Ohio, Missouri, and Iowa.

The *Scientific American Supplement* for August 5, 1916, contains an article by Professor R. E. MORITZ, of the University of Washington, describing the "Clyco-Harmonograph," an instrument for drawing in ink or pencil some sixty-three distinct species of mathematical curves, among them such known curves as the conchoids, the nephroids, the foliates, and harmonic curves. The method commonly employed in the construction of such curves is to plot the curve by points determined by their equations. This process is laborious, and accumulates inaccuracies. The Moritz instrument eliminates these errors, and constructs the curves with the greatest ease and precision. The article in the "Scientific American Supplement" shows a photograph of the instrument and reproductions of a number of harmonic curves constructed by it.

Recent papers read before the ASSOCIATION and the SOCIETY indicate that renewed interest is apparent in all phases of mathematical history. Hence, no apology is needed for the publication of notes such as the following:

In *Nature*, December 3, 1914, p. 363, Professor CAJORI showed that the cross $\times$ as a symbol of multiplication, which is said in histories to occur first in William Oughtred's *Clavis mathematicae* (1631), is given in form of the letter x and X in Edward Wright's translation of John Napier's *Mirifici logarithmorum canonis descriptio*, second edition, London, 1618, where we read, page 4: "The note of addition is (+), of subtracting (−), of multiplying ($\times$)." This is taken from a part of the book headed "Appendix to the Logarithmes," the authorship of which is not given but is believed now most probably to be attributed to William Oughtred.

In 1902 Professor W. W. BEMAN pointed out (*L'Intermédiaire des mathématiciens*, T. 9, Paris, p. 229, question 2424) that the colon (:) occurs as the symbol for geometric ratio at the end of the tables in Oughtred's *Trigonometria* of 1657. Professor CAJORI has found that the colon was so used by the astronomer Vincent

Wing in 1651, 1655 and 1656 and by a Suffolk schoolmaster with the initials "R. B." in 1655. For further details see *Nature*, Dec. 31, 1914, p. 477.

The first designation of the sides of a triangle by the same letters, respectively, as the angles opposite, one group of letters being capitals A, B, C, and the other group small letters a, b, c, has been attributed to Leonhard Euler (*Histoire de l'académie de Berlin, année*, 1753, p. 231), but Professor CAJORI finds that it occurs in a pamphlet containing trigonometric formulas published by Richard Rawlinson of Queen's College, Oxford, sometime between 1655 and 1668. Additional information on this is given in *Nature*, Feb. 11, 1915, pp. 642 and 643.

Recent discussions on the teaching of mathematics to students of engineering recall some earlier references to this topic. For instance, in the December, 1914, number of the *Bulletin* of the American Mathematical Society Mr. GEORGE PAASWELL, C.E., calls attention to the great need of applying fuller mathematical analysis to the problems of the applied science professions, and speaks of the appalling gaps in analysis which the engineer must bridge with assumptions far from rigorous or satisfactory; he appeals to producing mathematicians to turn their attention to bettering this state of affairs.

In the April 1915 number of the *Bulletin* Professor C. N. HASKINS makes reply to two leading points of criticism: (1) While the curricula of the schools of applied science are not sufficiently intensive or extensive to enable their graduates to meet these outstanding problems with a great wealth of mathematical power, the required mathematical courses represent very nearly the maximum of what can be effectively assimilated and used by the average student in such schools. Mathematically able students might however *elect* broader and deeper courses of a suitable sort. (2) As to the criticism that an engineer despairs of being able to read modern mathematical treatises unless in lines already familiar to him, it must be recognized that this "has its exact counterpart in the mathematician's despair of keeping up with modern engineering thought and practice." In analogy with courses leading to the degree of Doctor of Public Health, Professor Haskins suggests the establishing in a few advanced institutions of graduate courses wherein competent men may devote themselves to research in the problems of engineering and may at the same time prepare themselves for effective work in the mathematical problems of engineering.

SECOND ANNUAL MEETING OF THE ASSOCIATION.

The second annual meeting of the MATHEMATICAL ASSOCIATION OF AMERICA will be held at Columbia University in New York City on Thursday, Friday and Saturday, December 28, 29, 30, 1916. The meeting will open with a joint session on Thursday afternoon of the Association with the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science; at which time Professor

E. W. Brown, of Yale University, will deliver his retiring address as president of the American Mathematical Society, and Professor A. O. Leuschner will deliver his retiring address as vice-president of Section A of the American Association.

On Friday morning Professor Florian Cajori, of Colorado College, will give an address before the Association on the History of Fluxions; and Professor M. W. Haskell, of the University of California, will speak on University Courses in Mathematics intended for Teachers of Secondary Mathematics. The latter paper will be discussed by Professor J. W. Young of Dartmouth College and Professor Edward Kasner of Columbia University.

On Friday afternoon will be the meeting of Institutional Delegates devoted to a discussion of the subject of Mathematical Libraries for Colleges, including a report of the Library Committee by the Chairman, Professor W. B. Ford of the University of Michigan; and a paper by Dr. T. H. Gronwall of New York City on "A Nucleus for a Mathematical Library."

On Saturday morning Professor E. B. Wilson, of the Massachusetts Institute of Technology, will give an address on The Mathematics of Aerodynamics, and the subject will be discussed by Professor A. G. Webster of Clark University, and Professor E. W. Brown of Yale University.

On Thursday evening there will be a joint dinner of the Mathematical Association of America with the American Mathematical Society, and Section A of the American Association for the Advancement of Science.

On Friday between twelve and two o'clock there will be an exhibition of Portraits and Medals of Mathematicians from the collection of David Eugene Smith.

The annual business meeting, with election of officers and other important matters, will occur on Friday afternoon at half past three o'clock. All meetings of the ASSOCIATION will be held in room 301 of Hamilton Hall, Columbia University. The headquarters of both the Association and the Society will be at the Murray Hill Hotel. Members should make reservations at the earliest possible date.

The Program Committee, Professor David Eugene Smith, Chairman, stands as announced in the November issue. The Committee on Arrangements, Professor Thomas S. Fiske, Chairman, has been enlarged by the addition of Professor Joseph Bowden, of Adelphi College, Professor C. O. Gunther, of Stevens Institute, and Professor J. B. Chittenden, of the Brooklyn Polytechnic Institute.

The Committee on Libraries, whose preliminary report will be given at this meeting, was announced in the November MONTHLY, but through an oversight of the editors the names of two members were omitted. The full committee is as follows: Professor W. B. Ford of the University of Michigan, Chairman, Professor Florian Cajori of Colorado College, Professor E. S. Crawley of the University of Pennsylvania, Professor S. Lefschetz of the University of Kansas, Professor W. R. Longley of Yale University, and Professor R. E. Root of the United States Naval Academy.

INDEX TO VOLUME XXIII

FORMAL PAPERS

ARCHIBALD, R. C. Historical Note on Centers of Similitude of Circles, 159.
——— See under DISCUSSIONS.
ARMSTRONG, G. N. First Annual Meeting of the Ohio Section, 189.
BABBITT, A. See under DISCUSSIONS.
BAKER, R. P. See REVIEWS under Davisson.
BAUDIN, M. C. See under DISCUSSIONS.
BOHANNAN, R. D. Pascal Line Equations and Some Consequences, 194.
BRINKMAN, H. W. See under DISCUSSIONS.
BROWN, B. H. Centers of Similitude and their n-dimensional Analogies, 155.
BURGESS, H. T. On the Matrix Equation $BX = C$, 152.
——— A Tribute to Andrew Wheeler Phillips, 165.
BUSSEY, W. H. See REVIEWS under Leib.
BYERLY, W. E. Sailing to Windward, 368.
CAIRNS, W. D. The Mathematical Association of America, 1.
——— and Hedrick, E. R. First Summer Meeting of the Association, 273.
CAJORI, F. Napier's Logarithmic Concept: A Reply, 71.
——— See REVIEWS under Karpinski.
——— See under DISCUSSIONS.
CARMICHAEL, F. L. On the Diophantine Equation $x^4 + ay^4 = u^2 + bv^2$, 321.
CARSLAW, H. S. See under DISCUSSIONS.
CLEMENTS, G. R. See REVIEWS under Phillips.
COOLIDGE, J. L. A Simple Proof of Hart's Theorem, 14.
CURTISS, R. H. See REVIEWS under Weld.
DANTZIG, TOBIAS. Elementary Proof of a Theorem due to F. Morley, 246.
DEDERICK, L. S. Construction of an Algebraic Equation with an Irrational Root Approximately Equal to a Given Value, 69.
DINES, C. R. See under DISCUSSIONS.
DUVAL, E. P. R. See REVIEWS under Wentworth.
ELMENDORF, A. A Differentiating Machine, 292.
EMCH, A. Remarks on a Circle Theorem, 162.
FIELD, PETER, and ZIWET, ALEXANDER. The Accelerations of the Points of a Rigid Body, 371.
FISKE, T. S. Relations Between the Association and the Society, 296.
GLENN, O. E. See under DISCUSSIONS.
GRIFFIN, F. L. See under DISCUSSIONS.
HAZEN, J. V. Frank Asbury Sherman, 114.
HEDRICK, E. R. A Tentative Platform of the Association, 31.
——— The Mathematical Association of America, 1.
HORTON, GOLDIE. A Note on the Calculation of Euler's Constant, 73.
HORTON, GOLDIE. Concerning Roulettes, 237.
HOSKINS, L. M. "Quantity of Matter" in Dynamics, 34.
——— See REVIEWS under Miller.
HUNTINGTON, E. V. An Elementary Theory of the Exponential and Logarithmic Functions, 241.
INGIMUNDSON, B. See under DISCUSSIONS.
IRWIN, F. A Curious Convergent Series, 149.
IWERSON, R. See under DISCUSSIONS.
JAMES, G. A Note on the Sum of the Remainders of a Series, 333.
JOHNSON, R. A. A Circle Theorem, 161.
——— See under DISCUSSIONS.
KEYSER, C. J. A Tribute to John Howard Van Amringe, 15.
LEFSCHETZ, S. A Direct Proof of DeMoivre's Theorem, 366.
LEHMER, D. N. See REVIEWS under Carmichael.
LONGLEY, W. R. Use of Transcendental Equations in Analytic Geometry, 289.
LORAM, C. T. The Retirement of Sir Thomas Muir, 74.
LOVITT, W. V. Geometrical and Other Illustrations of Indeterminate Forms, 41
——— Preferential Voting, 363.
MILLER, G. A. A Tribute to Samuel Walker Shattuck, 45.
MITCHELL, U. G. See REVIEWS under Harding.
See under DISCUSSIONS.
MOORE, C. N. See under QUESTIONS.
MOULTON, E. J. The Real Function Defined by $X^y = Y^x$, 233.
NICHOLSON, J. W. See under DISCUSSIONS.
NYBERG, J. A. The Unification of Freshman Mathematics, 101.
PORTER, M. B. The Derivative of the Logarithm, 204.
PHILLIPS, H. B. See under DISCUSSIONS.
RICHARDSON, L. J. Digital Reckoning Among the Ancients, 7.
ROBBINS, R. B. See under QUESTIONS.
ROEVER, W. H. Graphical Constructions for a Function of a Function, and for a Function given by a Pair of Parametric Equations, 330.
——— See under QUESTIONS.
ROOT, R. E. See under DISCUSSIONS.
ROTH, W. E. A Simplified Proof of the Distributive Law of Multiplication Given in Hilbert's "The Foundations of Geometry," 202.
SIMPSON, T. M. See under DISCUSSIONS.
SLAUGHT, H. E. Editorial Notes on the Association, 29, 63, 93, 134, 183, 227, 360.
STAGER, H. W. Annual Meeting of the California Secondary School Teachers of Mathematics, 76.

SWIFT, E. See REVIEWS under Bôcher.
TYLER, H. W. See under DISCUSSIONS.
WEAVER, J. H. The Duplication Problem, 106.
WILCZYNSKI, E. J. See REVIEWS under Kiepert, Miller, Zoretti.
WILLIAMS, K. P. See under DISCUSSIONS.
WOODWARD, R. S. See under DISCUSSIONS.
ZIWET, ALEXANDER. See Field, Peter.

PROBLEMS, QUESTIONS, AND DISCUSSIONS

ADAMS, O. S. Problems proposed or solved in Algebra and Calculus, 253, 255, 303, 387, 389, 390, 392.
AGARD, H. L. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 49, 55, 59, 82, 86, 124, 215, 217, 218, 255, 258, 260, 304, 305, 308, 349, 389.
ALTSHILLER, N. Problems proposed or solved in Algebra and Geometry, 18, 80, 82, 84, 169, 253, 255, 301, 303, 341, 385, 387, 389.
ANNING, N. Problems proposed or solved in Algebra, Geometry, Calculus, and Number Theory, 19, 21, 50, 79, 120, 124, 169, 171, 172, 211, 213, 255, 258, 304, 346, 390.
ARCHIBALD, R. C. Discussion of Courses for the Training of Teachers of Mathematics, 397.
ATCHINSON, C. S. Problem solved in Calculus, 390.
BABBITT, A. Two useful relations in Trigonometry, 133.
BAKER, R. P. Problems proposed in Geometry and Number Theory, 220, 341.
BALCH, J. V. Problems solved in Algebra and Calculus, 82, 218.
BAREIS, GRACE M. Problems solved in Algebra and Calculus, 49, 349.
BARNETT, I. A. Problems solved in Calculus and Number Theory, 255, 349.
BARNHART, C. A. Problem solved in Algebra, 303, 390.
BARROW, D. F. Problem proposed in Geometry, 210.
BAUDIN, M. C. Discussion of the folium of Descartes, 90.
BELL, E. T. Problems proposed in Algebra, Geometry, and Number Theory, 49, 121, 122, 209, 210, 385.
BEMAN, W. W. Problems solved in Calculus, 59, 258.
BENNETT, A. A. Problem proposed in Number Theory, 210.
BERGSTRESSER, C. A. Problem solved in Calculus, 218.
BLANCHARD, G. Problem solved in Algebra, 213.
BOHANNAN, R. D. Problem solved in Algebra, 213.
BOND, J. D. Problem solved in Calculus, 390.
BOWERMAN, M. R. Problem solved in Mechanics, 392.
BRACHO, J. F. Problem solved in Calculus, 23.
BRINKMAN, H. W. Interpretation of Degenerate Conics, 132.
BROWN, B. J. Problems proposed or solved in Algebra, Calculus, and Mechanics, 55, 59, 80, 81, 128, 169, 178, 258.
BROWN, E. W. Problem proposed in Algebra, 341.
BROWN, H. S. Problems solved in Algebra, 212, 255.
BULLARD, J. A. Problems proposed or solved in Algebra, Geometry, and Calculus, 210, 218, 254, 255, 259, 304, 305, 349, 389, 390.
BURTON, W. W. Problems proposed or solved in Algebra and Calculus, 86, 124, 217.
CAIRNS, W. D. Problems proposed or solved in Algebra, 18, 48, 49, 212.
CAJORI, F. Problem proposed in Geometry, 82. Comment on Criticism by H. S. Carslaw, 312.
CANADAY, E. F. Problems solved in Algebra, 213, 303, 387.
CAPARO, J. A. Problems solved in Algebra, Geometry, Calculus, and Mechanics, 23, 55, 82, 83, 86, 88, 128, 129, 172, 173, 216, 217, 218, 255, 260, 261, 344, 389, 390.
CAPRON, P. Problems proposed or solved in Algebra, Geometry, Calculus, and Mechanics, 20, 47, 55, 83, 87, 119, 124, 129, 169, 211, 216, 253, 255, 258, 261, 303, 304, 305, 308, 342, 346, 348, 387, 389, 390, 392, 393.
CARLETON, H. N. Problems proposed or solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 82, 84, 86, 128, 129, 260, 261, 302, 303, 346, 353, 387.
CARMICHAEL, R. D. Problems proposed in Number Theory, 121, 122, 261, 309.
CARSLAW, H. S. Discussion of Napier's Logarithmic Concept, 310.
CARTER, MARY. Problem solved in Number Theory, 394.
CARRUTH, W. M. Problem solved in Calculus, 390.
CLARK, L. Problem solved in Calculus, 218.
CLARKE, E. H. Problems proposed in Calculus and Number Theory, 54, 129, 218.
CLARKE, J. A. Problem proposed in Geometry, 84.
CLAWSON, J. W. Problems proposed or solved in Algebra, Geometry, and Calculus, 23, 50, 54, 82, 83, 84, 86, 214, 302, 303, 346, 385, 389, 390.
COFFIN, L. M. Problem solved in Calculus, 390.
COOLIDGE, J. L. A Correction, 92.
CROMWELL, J. W. Problems solved in Geometry and Calculus, 346, 390.
CULLUM, W. Problems proposed and solved in Calculus, 56, 59.
DANTZIG, T. Problem solved in Calculus, 219.
DAVIS, MRS. E. B. Problem solved in Number Theory, 220.
DAVIS, H. T., Problems proposed or solved in Geometry, Mechanics, and Number Theory, 80, 172, 342.

DE LONG, I. M. Problem proposed in Algebra, 300.
DICKSON, L. E. Problem proposed in Number Theory, 121.
DIMICK, C. E. Problems solved in Geometry and Calculus, 304, 305, 349.
DINES, C. R. Theorem on a Number Relation, 89.
DRANE, W. H. Problem proposed in Algebra, 20.
EELLS, W. C. Problems proposed or solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 21, 23, 49, 54, 87, 217, 305, 308, 353, 387.
EPPERSON, C. A. Problem solved in Geometry, 304.
ESCOTT, E. B. Problems proposed or solved in Algebra, 213, 214, 300.
FEEMSTER, H. C. Problems proposed or solved in Algebra, Geometry, Calculus, and Mechanics, 49, 82, 86, 129, 173, 217, 260, 302, 303, 341, 343, 387, 388, 389, 392.
FINKEL, B. F. Problems proposed or solved in Calculus and Mechanics, 23, 51, 53, 305, 342, 349.
FLANAGAN, C. E. Problem solved in Calculus, 86.
FLEMING, W. F. Problem proposed in Geometry, 85.
FORDERO, F. (Seville, Spain). Problem solved in Calculus, 86.
GAINS, R. E. Problem solved in Algebra, 213.
GIBSON, EMMA. Problems proposed or solved in Calculus and Mechanics, 59, 128, 129, 177.
GINSBURG, J. J. Problems proposed or solved in Algebra, 209, 255, 386, 387.
GLENN, O. E. Discussion of quadratic factors of a polynomial, 313; Discussion of the number of terms between two given terms of an ordered polynomial, 355.
GREENSTREET, W. J. Problem proposed in Calculus, 385.
GRIFFIN, F. L. Discussion of unification of freshman mathematics, 223; Problems solved in Algebra, Geometry and Calculus, 49, 57, 82, 83.
GRONWALL, T. H. Criticisms of solutions in Calculus and Number Theory, 306, 309; Problem proposed in Algebra, 341.
HALDEMAN, C. B. Problem proposed in Algebra, 300.
HANSON, H. O. Problem proposed in Algebra, 253.
HARDING, A. M. Problems solved in Algebra, Calculus and Mechanics, 23, 81, 87, 124, 125, 128, 387, 390.
HARTWELL, G. W. Problems solved in Algebra, Geometry, and Calculus, 59, 82, 84, 86, 124, 215, 217, 218, 254, 303, 304, 346, 386, 388, 389, 390.
HEAL, W. E. Problem solved in Algebra, 81.
HOLMES, A. H. Problems proposed or solved in Algebra, Geometry, Calculus, and Number Theory, 22, 83, 122, 125, 127, 172, 217, 255, 262, 303, 305.
HOOVER, WILLIAM. Problem solved in Mechanics, 393.
HORNE, C. E. Problems solved in Algebra, Geometry, and Calculus, 49, 82, 172, 217, 258.
HORNUNG, C. Problem solved in Calculus, 218.
HOWARD, H. R. Problems solved in Algebra and Geometry, 255, 304.
HOWARD, T. Problems solved in Calculus and Mechanics, 258, 308.
HUTCHINSON, C. A. Problem solved in Calculus, 390.
INGIMUNDSON, B. The mechanical trisection of an angle, 181.
INGVALE, A. Problems proposed or solved in Algebra and Geometry, 119, 388.
IRWIN, F. Problems proposed or solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 18, 48, 49, 84, 86, 131, 168, 172, 174, 175, 213, 220, 259, 310, 344, 352, 385, 393.
IWERSON, R. An approximate construction for an ellipse, 354.
JOFFE, S. A. Problems proposed or solved in Algebra and calculus, 23, 85, 170.
JOHNSON, R. A. Discussion of the Simson line, 61; Correction in solution of Geometry problem, 214; Problems proposed or solved in Geometry, 120, 215, 304.
JOHNSON, W. W. Problems solved in Geometry, 173, 304.
KATANICK, H. R. Problems proposed in Number Theory, 301, 353.
KEMPNER, A. J. Problems proposed in Algebra and Number Theory, 79, 122, 302, 310.
KENDALL, C. Problems proposed or solved in Algebra and Calculus, 305, 390.
KENYON, A. M. Problems proposed or solved in Algebra, 47, 48, 122, 124, 213.
KEULIGAN, G. Problem solved in Calculus, 59.
KLAUFF, I. Problem solved in Mechanics, 309.
LEFSCHETZ, S. Reply to a question on the solution of an equation, 26.
LEWIS, F. P. Problems proposed or solved in Geometry, 210, 257.
LORD, R. W. Problem solved in Geometry, 304.
LOUD, F. H. Problem solved in Algebra, 388.
MACAULAY, C. Problem proposed in Number Theory, 121.
MCGREGOR, R. T. Problem proposed in Geometry, 341.
MARTIN, ARTEMAS. Problem proposed in Number Theory, 121.
MASON, T. E. Problems proposed in Number Theory, 122, 352.
MATHEWS, R. M. Problems solved in Algebra, 87, 124, 387.
MATHEWSON, L. C. Problems solved in Algebra and Number Theory, 130, 214.
MILLS, C. N. Problems proposed or solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 18, 19, 79, 80, 81, 82, 86, 87, 120, 124, 127, 129, 168, 169, 171, 210, 215, 217, 218, 253, 254, 257, 301, 302, 308, 385.

MILNE, W. E. Problem solved in Algebra, 255.
MISER, W. L. Problem proposed in Calculus, 173.
MITCHELL, U. G. Discussion of Courses for the Training of Teachers of Mathematics, 395.
MOORE, C. N. Reply to question on the value of Mathematics in the high school, 60.
MOOTS, E. E. Problems proposed in Geometry and Mechanics, 169, 301.
MORGAN, F. M. Problems solved in Geometry and Calculus, 304, 349.
MORLEY, F. V. Problem proposed in Geometry, 210.
MORLEY, R. K. Problem solved in Calculus, 346.
MORRIS, F. R. Problems proposed or solved in Algebra, Geometry, Calculus, and Mechanics, 80, 82, 86, 124, 125, 173, 216, 260, 301, 303, 307, 346.
MOULTON, E. J. Problems proposed or solved in Algebra, Mechanics, and Number Theory, 80, 301, 302.
NICHOLS, I. C. Problem solved in Algebra, 49, 81.
NICHOLSON, J. W. A question with respect to 0/0, 180; Problem proposed in Number Theory, 80.
NICKLE, C. A. Problems solved in Algebra, 255, 389.
OLSON, H. Problems proposed or solved in Algebra, Geometry, Calculus, Mechanics, and Number Theory, 22, 49, 55, 59, 82, 125, 129, 131, 175, 179, 217, 219, 255, 260, 302, 303, 304, 308, 309, 342, 344, 346, 353, 387, 389, 390, 392.
OVIN, ELIZABETH. Problem solved in Calculus, 86.
PAASWELL, G. Problems proposed or solved in Geometry and Calculus, 301, 305, 346, 387, 389, 390, 392.
PANDYA, N. P. (Sojitra, India). Problems proposed or solved in Algebra and Geometry, 20, 83, 84, 86, 169, 253, 387.
PHILLIPS, H. B. Discussion of slope-angle of a curve, 224; Problems proposed in Geometry and Calculus, 119, 174.
PICKETT, L. M. Problem solved in Calculus, 218.
POLISH, H. Problems solved in Algebra, Calculus and Mechanics, 59, 124, 261.
POTTLE, F. A. Problem solved in Calculus, 389.
RASOR, S. E. Problems solved in Algebra, Geometry and Calculus, 59, 123, 218, 304.
RAU, A. G. Problem solved in Calculus, 389, 390.
RAYNOR, G. Problem solved in Calculus, 218.
RAYWORTH, J. C. Problem solved in Calculus, 218.
REYNOLDS, J. B. Problems proposed or solved in Algebra, Calculus, and Mechanics, 86, 88, 119, 210.
RIGGS, W. F. Problem solved in Algebra, 303.
RILEY, J. L. Problems solved in Calculus and Mechanics, 258, 308.
ROBBINS, C. K. Problems solved in Algebra, Geometry, and Calculus, 255, 304, 305, 349.
ROBBINS, R. B. Reply to question on college courses, 24.
ROEVER, W. H. Reply to question on Descriptive Geometry, 266.
ROOT, R. E. Economic value of the Mathematical Association of America, 63; Problem solved in Mechanics, 393.
ROWE, J. E. Problems proposed or solved in Algebra and Number Theory, 47, 123, 254.
RUMBLE, D. Problem solved in Calculus, 217.
SCHMALL, C. N. Problems proposed or solved in Algebra, Geometry, Calculus, and Mechanics, 20, 23, 47, 55, 79, 80, 84, 120, 124, 168, 169, 172, 218, 219, 253, 258, 259, 260, 301, 303, 306, 308, 349, 385, 392.
SCHUYLER, E. Problems proposed or solved in Geometry, Calculus, and Mechanics, 59, 83, 215, 218, 303, 308.
SCHWEITZER, A. R. Problem proposed in Number Theory, 121.
SIMPSON, T. M. Discussion of the teaching of logarithms, 264.
SINITSSCY, BENJAMIN. Problem solved in Algebra, 387.
SMITH, A. W. Problems solved in Algebra and Calculus, 303, 348, 383, 390.
SOSNOW, G. Y. Problem proposed in Algebra, 253.
SOUSLEY, C. P. Problem solved in Calculus, 390.
SPARROW, C. M. Problem solved in Geometry, 304.
SPUNAR, V. M. Problem proposed and solved in Calculus, 258.
STROM, C. A. W. Problem solved in Algebra, 124.
SWIFT, E. Problems proposed or solved in Algebra, Geometry, Calculus, and Mechanics, 21, 48, 59, 124, 131, 172, 175, 211, 213, 214, 218, 257, 258, 261, 262, 303, 304, 308, 348, 387, 390, 393.
THOMAS, R. G. Problems proposed in Geometry, 47, 304.
THOME, W. J. Problems solved in Algebra and Geometry, 303, 305, 387, 392.
TYLER, H. W. The Mathematical Association of America and school problems, 96.
UHLER, H. S. Problems proposed or solved in Algebra, Geometry, Calculus, and Mechanics, 19, 47, 49, 51, 59, 85, 86, 88, 120, 124, 125, 171, 174, 176, 177, 178, 209, 217, 254, 255, 260, 303, 304, 305, 308, 309, 342, 346, 350, 388, 392.
VANCE, E. H. Problem solved in Algebra, 255.
WAGAR, G. L. Problem solved in Algebra, 303, 387.
WALTON, T. O. Problem solved in Geometry, 304.
WARREN, L. A. H. Problem solved in Algebra, 387.
WEAVER, J. H. Problem solved in Algebra, 124; A geometric illustration of the form ∞/∞, 181.

WELD, L. G. Problems proposed or solved in Geometry, Calculus, and Mechanics, 19, 79, 176, 214, 216, 218, 260, 307, 344, 346, 348.
WHITFORD, E. E. Problems solved in Algebra, 213, 303.
WHITTED, J. A. Problem solved in Calculus, 218.
WILLIAMS, K. P. Discussion of some determinants connected with the Bernoulli numbers, 263.
WILLIAMS, W. H. Problem solved in Mechanics, 127.
WILSON, A. H. Problems solved in Algebra, Calculus, and Mechanics, 124, 218, 261, 348.
WILSON, E. B. Problem solved in Algebra, 255.
WITHERS, S. C. Problem solved in Algebra, 214.
WOOD, FREDERICK. Problems solved in Algebra and Calculus, 59, 390.
WOODWARD, R. S. Congratulations on the formation of the Mathematical Association of America, 95.
WORTHINGTON, E. H. Problem solved in Algebra, 387.
YEN, C. C. Problems proposed and solved in Calculus, 385, 387.

BOOK REVIEWS

Bôcher, M. Plane Analytic Geometry. E. SWIFT, 17.
Cantor, Georg. Contributions to the Founding of the Theory of Transfinite Numbers. T. H. HILDEBRANDT.
Carmichael, R. D. Diophantine Analysis. D. N. LEHMER, 166.
Davison, C. Subjects for Mathematical Essays. R. P. BAKER, 298.
Harding, A. M. and Turner, J. S. Plane Trigonometry with Tables. U. G. MITCHELL, 250.
Karpinski, L. Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi with an Introduction, Critical Notes, and an English Version. F. CAJORI, 116.
KIEPERT, L. Grundriss der Differential- und Integral-Rechnung. E. J. WILCZYNSKI, 206.
Leib, D. D. Problems in the Calculus with Formulas and Suggestions. W. H. BUSSEY, 78.
Lilly, S. B. See Miller, J. A.
Mancini, Gerolamo. The Francesca Manuscript, "De Corporibus Regularibus." FLORIAN CAJORI.
Miller, G. A. Historical Introduction to Mathematical Literature. E. J. WILCZYNSKI, 207.
Miller, J. A. and Lilly, S. B. Analytic Mechanics. L. M. HOSKINS, 337.
Phillips, H. B. Analytic Geometry. G. R. CLEMENTS, 117.
Richardson, Sophia F. Solid Geometry. LENNIE P. COPELAND.
Smith, D. E. See Wentworth.
Turner, J. S. See Harding.
Weld, L. D. Theory of Errors and Least Squares. R. H. CURTISS, 249.
Wentworth, G., and Smith, D. E. Plane and Spherical Trigonometry with Tables. E. P. R. DUVAL, 299.
Zoretti, L. Leçons de Mathématiques Générales. E. J. WILCZYNSKI, 207.

NEW BOOKS, BRIEF MENTION

Archibald, R. C. Euclid's Book on Division of Figures, 146.
Betz, W., and Webb, H. E. Solid Geometry, 206.
——— Plane and Solid Geometry, 298.
Blichfeldt, H. F. See Miller.
Breslich, E. R. First Year Mathematics for Secondary Schools, 116.
Brown, J. C. See Finney.
Bureau of Census, Life Tables, 231.
Byerly, W. E. An Introduction to the Use of Generalized Coördinates in Mechanics and Physics, 297.
Cantor, G. Contributions to the Founding of the Theory of Transfinite Numbers. Translated by P. E. B. Jourdain, 115.
Chappell, E. Five-Figure Mathematical Tables, 206.
Crathorne, A. R. See Rietz.
De Morgan, Augustus. A Budget of Paradoxes, 115.
Dickson, L. E. See Miller.
Dunkel, O. See Hedrick.
Dupuis, N. F. Elementary Synthetic Geometry of the Point Line and Circle, 97.
——— Elementary Synthetic Solid Geometry, 97.
Durell, F. Fundamental Sources of Efficiency, 116.
Encyclopédie des Sciences Mathématiques, 27.
Finney, H. A., and Brown, J. C. Modern Business Arithmetic, 298.
Flexner, A. A Modern School, 186.
Ford, W. B. Studies on Divergent Series and Summability, 186.
Glenn, O. E. The Theory of Invariants, 26, 65.
Hart, W. W. See Wells.
Hedrick, E. R. Constructive Geometry, 187.
——— and Dunkel, O. Goursat's Mathematical Analysis. Vol. II. Part I. Functions of a Complex Variable, 206.
Hunt, Brenelle. A Community Arithmetic, 206.
Jackson, C. S., Whipple, F. J. W., and Roberts, Lucy. A Twentieth Century Arithmetic, 116.

Jourdain, P. E. B. See Cantor.
Karapetoff, V. Engineering Applications of Higher Mathematics, 297.
Landis, E. H. See Richardson.
Mann, H. L. A Text Book on Practical Mathematics, 97.
Martin, L. A. A Text Book of Mechanics, Vol. VI, 206.
Miller, G. A., Blichfeldt, H. F., and Dickson, L. E. Theory and Application of Finite Groups, 65, 98, 115.
Napier Tercentenary Memorial Volume, 400.
Palmer, C. G., and Taylor, D. P. Plane Geometry, 146.
Popplewell, W. C. The Elements of Surveying and Geodesy, 115.
Richardson, R. P. Fundamental Conceptions of Modern Mathematics, 115.
Rietz, H. L., Crathorne, A. R., and Taylor, E. H. School Algebra, First and Second Course, 116.
Roberts, Lucy. See Jackson.
Sandwick, R. How to Study and What to Study, 116.
Schwartz, A. J. See Young.
Smith, D. E. See Wentworth.
Taylor, D. P. See Palmer.
Taylor, E. H. See Rietz.
Townsend, E. J. Functions of a Complex Variable, 115.
Tracy, J. I. See Wilson.
Vivanti, G. Elementi della Teoria delle Equazioni Integrali Lineari, 358.
Webb, H. E. See Betz.
Webster, A. G. Partial Differential Equations of Mathematical Physics, 97.
Wells, W., and Hart, W. W. Plane and Solid Geometry, 116.
Wentworth, G., and Smith, D. E. Trigonometric and Logarithmic Tables, 116, 297.
Whipple, F. J. W. See Jackson.
Whittaker, E. T. A Course of Modern Analysis, 146.
Williams, J. H., and Williams, K. P. Solid Geometry, 146, 206.
Wilson, W. A. Analytic Geometry, 206.
Young, J. W., and Schwartz, A. J. Plane Geometry, 116.

PERSONAL MENTION

Aley, R. J., 401; Argard, H. L., 97; Allen, R. B., 190, 192; Altshiller, N., 316; Anderegg, F., 191, 193; Archibald, R. C., 184, 229, 288; Armstrong, G. N., 191; Arnette, W., 28.

Babbitt, A., 316; Barrow, D. F., 400; Beal, F. W., 317; Beetle, R. D., 399; Bell, E. T., 97; Beman, W. W., 402; Bennett, A. A., 316, 317, 358; Birkhoff, G. D., 319; Blanchard, R. L., 229; Bliss, G. A., 230; Böcher, M., 230, 317; Börger, R. L., 357; Brasefield, E., 400; Brown, B. H., 229; Brown, E. B., 67; Bullard, J. A., 399; Burgess, H. T., 358; Burgess, R. W., 147, 184, 316; Burgess, W., 229; Burnside, W., 146.

Cain, William, 400; Cairns, W. D., 277, 281, 315; Cajori, F., 66, 402, 404; Canaday, E. F., 269; Carslaw, H. S., 317; Carty, J. J., 229; Clevenger, H. C., 357; Cobb, H. R., 146; Cole, F. N., 319; Collins, J. V., 67; Colpitts, E. C., 97; Colpitts, M., 316; Conant, L. L., 401; Coolidge, J. L., 276, 284, 319, 358; Coon, Edith M., 400; Cowley, E. B., 269; Crathorne, A. R., 27, 283; Currier, C. H., 259.

Dalman, M. A., 401; Davis, J. M., 97; Davisson, S. C., 402; Dedekind, J. W. R., 145; Deming, H. G., 184; Dickey, R. W., 269; Dines, L. L., 317; Doan, F. C., 230; Dodd, E. L., 270; Dresden, A., 68, 229; Duncan, J. C., 317; Dunkel, O., 357; Durell, F., 65; Dustheimer, O. L., 230.

Echegaray, Don José, 399; Eells, W. C., 269; Eisenhart, L. P., 284, 358; Eliot, C. W., 146; Emch, A., 145, 147, 358; Emmons, L. C., 269; Escott, E. B., 357; Evans, G. C., 320.

Faught, J. B., 190, 192; Field, P., 26; Fields, J. C., 146; Fine, H. B., 28, 230, 276, 277, 319; Fiske, T. S., 276, 319; Fite, W. B., 284; Flexner, Abraham, 401; Florance, E., 230, 401; Focke, T. M., 191, 193; Ford, W. B., 68; Forsyth, C. H., 28; Forsyth, F. M., 399; Fort, T., 26.

Garabedian, C. A., 316; Garlaugh, C., 145; Garner, H. M., 269; Gilman, R. E., 228, 316; Gist, A. H., 316; Gittleman, L. B., 401; Glazier, H., 190, 192; Gleason, 316; Glover, J. W., 231; Godeau, L., 66; Goldsborough, C. R., 146; Graber, M. E., 190, 191; Graf, J. H., 66; Grant, E. D., 318; Grennan, E. B., 316; Griffin, F. L., 277; Gronwall, T. H., 229; Gunther, C. O., 230, 359.

Hale, G. E., 229; Harding, A. M., 316; Hardy, G. H., 146; Harris, E., 358; Hart, W. L., 184, 357; Haskell, M. W., 228, 276, 284; Haskins, C. N., 316, 400; Hassler, J. O., 358; Hawkesworth, A., 358; Hayes, E., 269; Hazlett, O. C., 185; Hedrick, E. R., 67, 99, 185, 228, 319; Henderson, Archibald, 400; Hildebrandt, T. H., 26; Hill, J. H., 269; Hodge, F. H., 401, 402; Holgate, T. F., 358; Holmes, A. H., 399; Hoover, W., 191, 193; Hopkins, M. B., 146; Hornley, P. W., 28; Hornung, C., 191, 192; Hoskins, L. M., 277, 278; Huntington, E. V., 145, 229, 272, 277, 288; Hurwitz, W. A., 284; Hyashi, T., 358.

Ingham, E. S., 230, 401; Ingham, J. A., 400.

Jackson, D., 145, 317; Jacobs, J., 28; James, V., 269; John, J. P. D., 317; Johnson, E. N., 401; Johnson, W. W., 319; Jordan, H. E., 28; Jourdain, P. E. B., 187, 400.

Karpinski, L. C., 26, 98, 400; Kellogg, O. D., 147; Kenyon, A. M., 65; Keyser, C. J., 76, 228; Kircher, E., 184; Klein, J. R., 358; Kryjanovsky, D., 66.

Larew, G., 318; Larsen, A. W., 28; Lasley, J. W., Jr., 400; Lazzeri, G., 66; Lefschetz, S., 27, 28, 317; Leib, D. D., 400; Le Valley, R. T., 28; Levy, C. T., 228; Lewis, A. D., 97;

Locke, L. L., 316; Longley, W. R., 317; Lord, W. L., 269; Lorey, W., 231; Love, C. E., 147.

McCain, G. I., 65; McClintock, E., 271; Macham, E. D., 228; McKim, F. J., 269; McMackin, F. J., 269; MacNeish, H. F., 400; Maglott, E. S., 316; Martin, A., 147; Mathews, R. M., 28; Mathewson, L. C., 147; Mendenhall, W. O., 402; Mercer, G. E., 359; Merrill, A. S., 228, 318; Merrill, H. A., 276; Metzler, W. H., 271; Miles, E. J., 185; Miller, A., 65; Miller, B. I., 97; Miller, E. B., 317; Miller, G. A., 28, 65, 67; Miller, J. A., 270; Millis, J. F., 27; Mitchell, U. G., 27, 28, 164; Mittag-Leffler, M. G., 185, 270, 276; Moore, C. N., 27, 66, 190, 192, 270; Moore, E. H., 229, 283, 288; Morgan, F. M., 399; Moritz, R. E., 402; Morley, F., 246; Morris, C. C., 190, 191, 230; Morris, F. R., 359; Morris, R., 359, 400; Moulton, F. R., 230; Myers, H. W., 316.

Nelson, A. L., 27; Nelson, C., 28; Neville, E. H., 317, 358; Newson, M., 165; Noble, C. A., 359; Norton, A. H., 228; Norwood, C. E., 399; Nowlan, F. S., 269, 399.

Oglesby, E. J., 399; Olds, G. D., 185, 276; Orner, G. D., 28; Osborne, C. P., 400.

Painlevé, P., 26; Paolina, Q., 66; Papps, P. C. H., 28; Pascal, E., 269; Pearson, K., 358; Phillips, A. W., 165; Pfeiffer, G. A., 228; Pitcher, A. D., 147, 230; Porter, M. B., 270.

Ramsey, J. B., 28; Raper, L. W., 65; Rasor, S. E., 190, 191; Reynolds, C. M., 316; Rice, C. D., 270; Richardson, R. G. D., 229; Richardson, S. F., 145; Reitz, H. L., 27, 68, 146; Rider, P. R., 400; Ritt, J. F., 358; Ritter, A., 28; Roever, W. H., 358; Rothrock, D. A., 401; Rowe, J. E., 399; Rugg, H. O., 27; Running, T. R., 26; Russell, B., 317.

Sarton, G., 228; Scheidenberger, F., 28; Schorling, R., 28; Shaw, J. B., 277, 284; Sheffer, H. M., 184, 400; Sherman, F. A., 114; Shinn, C., 28; Skimming, W. B., 230; Slaught, H. E., 276; Smart, W. M., 270; Smith, D. E., 147, 272, 277, 279, 283, 319; Smith, F. L., 316; Smith, G. W., 28; Snedden, David, 401; Snyder, E. R., 359; Snyder, V., 67; Sperry, P., 146; Stager, H. W., 359; Stark, M. E., 229; Stouffer, E. B., 28; Stein, G., 147; Steimley, L. L., 28; Stone, J. C., 230; Stone, R. B., 401; Stone, W. B., 401; Strang, B. B., 28; Stromquist, C. E., 65.

Tanzola, J. J., 399; Taylor, E. H., 27; Thompson, G. P., 269; Todd, A. J., 65; Torrey, M. M., 229; Tracey, G. I., 269; Tweedie, C., 65; Twiss, G. R., 230; Tyler, H. W., 276, 283, 360.

Van der Vries, J. N., 165, 272, 277, 279; Van Vleck, E. B., 229; Veblen, O., 229, 283, 288, 320, 359; Vivanti, G., 66.

Webb, H. E., 359; Webster, A. G., 319; Wells, W., 229, 360; West, A., 28; Wheeler, J. J., 317; White, H. S., 67, 246; Whittemore, J. K., 400; Wilczynski, E. J., 67, 359; Wiley, F. B., 191; Williams, K. P., 317; Wilson, E. B., 229; Winger, R. M., 147; Wood, F. E., 316; Woods, B. M., 76; Woods, F. S., 272, 277, 281.

Yanney, B. F., 190, 192; Young, A. E., 190, 191; Young, J. W. A., 186, 277, 283; Young, W. H., 146.

Zeis, H. C., 27; Ziwet, A., 276, 288.

JOURNALS, ASSOCIATIONS, ETC.

Acta Mathematica, 185, 270. American Association of University Professors, 99. American Journal of Mathematics, 147, 185. American Mathematical Society, 66, 184, 188, 227, 319; Chicago Section, 67, 98, 185; Colloquium, 319, 402. Annals of Mathematics, 288, 317, 318, 358. Association of Mathematics Teachers of New Jersey, 230, 359. Association of Ohio Teachers of Mathematics, 230. Association of Teachers of Mathematics in the Middle States and Maryland, Pittsburgh Section, 98.

Bolletino di Bibliographia e Storia delle Scienze Matematiche, 66. Bulletin of the University of Kansas, 27. Bureau of Education Bulletin, 27.

California High School Teachers Association, 359. Cambridge Tracts in Mathematics, 146. Educational Times, 100.

International Congress of Mathematicians, 270.

Journal of Educational Psychology, 146.

L'Enseignement Mathématique, 67.

Mathematical Association of America, Charter Members, 93, 134, 271; Committees, 95, 96, 183, 226, 361; Library, 64, 183; Organization and Constitution, 1; Council Announcements, 225; Meetings, 29, 63, 183, 227, 272, 273, 318, 360, 404; Platform, 31; Progress, 93; Indiana Section, 402; Iowa Section, 187, 270, 361; Kansas Section, 164, 361; Missouri Section, 361; Ohio Section, 185, 189, 361. Mathematical Clubs, 28, 98, 99, 147, 185, 186, 187, 188, 230, 318, 358. Mathematical Gazette, 65, 317, 358. Mathematics Section of the Indiana State Teachers' Association, 402. Mathematics Teacher, 147. Mind, 187. Monist, 400.

Naturae Novitatis, 185.

Periodico di Matematica, 66. Philosophical Transactions of the Royal Society of London, 358. Proceedings of the Edinburgh Mathematical Society, 186. Proceedings of the London Mathematical Society, 146.

Rendiconti della Accademia dei Lincei, 269. Rendiconti del Circolo Matematico di Palermo, 358. Revista de Matematicas, 186.

School and Society, 65, 146, 185, 186. School Science and Mathematics, 28, 65. Science, 27, 66, 99. Scientific American, 402. Scientific Monthly, 67. Scientific Series of Publications of the University of Michigan, 186. Summer Session Courses, 148, 231.

Texas Mathematics Teachers' Bulletin, 270.

Washington University Studies, 358.

Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, 231.

ERRATA NOTED IN VOLUME XXIII.

Page 53, in the figure the point $M.$ should be symmetrical with the point $k.$
Page 54, line 3, R before the integral sign should be $R^2.$
Page 54, line 9, the denominator of the last term should be "mR" instead of "$\sqrt{R^2 - b^2}$."
Page 54, line 11, the first parenthesis should precede "tan."
Page 160, first line of note 2, for "collectionis" read "collectio."
Page 161, line 14, add "or internal" after "external."
Page 161, line 15, add "or external" after "internal."
Page 161, line 3 of footnote 1, for "Elucid" read "Euclid."
Page 304, line 16, for "radical axis" read "given line."
Page 329, line 18, for "$u = 1$" read "$m = 1$."
Page 334, line 18, for "$(n - 1)$th term" read "the term a_{n-1}."
Page 353, Example 228, for "Hermon C." read "Herman R."
Page 354, line 14, change $y = 8$ to $y = 5.$
Line 2 up, for "Ox" read "OY."
In the diagram change B' to $B.$
Page 360, line 22, for "first" read "second."

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H
HH
H

INDEX TO VOLUME XXIV.

FORMAL PAPERS—AUTHORS.

ALLEN, E. S. See REVIEWS under Carslaw.
ARCHIBALD, R. C. Frère Gabriel Marie, 280.
BRADSHAW, J. W. See under DISCUSSIONS.
BUSSEY, W. H. The Origin of Mathematical Induction, 199.
——— See REVIEWS under Lebmer.
CAIRNS, W. D. Second Annual Meeting of the Association, 49.
——— Second Summer Meeting of the Mathematical Association of America, 353.
CAJORI, F. Discussion of Fluxions: from Berkeley to Woodhouse, 145.
CAPRON, P. See under DISCUSSIONS.
CHATBURN, G. R. See REVIEWS under Mills.
CLAWSON, J. W. An Inversion of the Complete Quadrilateral, 71.
COOLIDGE, J. L. A Simple Geometrical Paradox, 76.
COREY, S. A. See under DISCUSSIONS.
CRATHORNE, A. R. See Mitchell.
DALAKER, H. H. See REVIEWS under Townsend.
DANIELL, P. J. New Rules of Quadrature, 109, 302.
DAVIS, ELIZABETH B. See under DISCUSSIONS.
DINES, L. L. Concerning Preferential Voting, 321.
DUNKEL, O. See under DISCUSSIONS.
EMCH, A. A Problem in Perspective, 379.
——— A Useful Principle in Curve Tracing, 168.
——— See REVIEWS under Coolidge.
EVANS, G. W. Cavalieri's Theorem in his own Words, 447.
FORD, W. B. (Chairman of Library Committee). A List of Mathematical Books for Schools and Colleges, 368.
——— See REVIEWS under Byerly.
FRANKLIN, P. See under DISCUSSIONS.
FRUMVELLER, A. F. Quadratic Factors of Polynomials, 208.
——— The Graph of $F(x)$ for Complex Number, 409.
GARRETT, W. H. See Mitchell.
GRAHAM, B. Some Calculus Suggestions by a Student, 265.
HEAL, W. E. See under DISCUSSIONS.
HEDRICK, E. R. The Significance of Mathematics, 401.
HILL, L. S. See under DISCUSSIONS.
—— See REVIEWS under Huntington.
HODGE, F. H. See under DISCUSSIONS.
HOWLAND, L. A. See under DISCUSSIONS.
HUNTINGTON, E. V. On Setting up a Definite Integral without the use of Duhamel's Theorem, 271.
——— The Logical Skeleton of Elementary Dynamics, I.
——— See under DISCUSSIONS.
JACKSON, C. S. A Problem in Probability, 73.
JACKSON, D. See under DISCUSSIONS.
JOHNSON, R. A. Directed Angles and Inversion, with a Proof of Schoute's Theorem, 313.
——— Directed Angles in Elementary Geometry, 101.
——— See under DISCUSSIONS.
KARPINSKI, L. Algebraical Developments among the Egyptians and Babylonians, 257.
——— See REVIEWS under Cajori.
KEMPNER, A. Simple Hints on Plotting Graphs in Analytic Geometry, 17.
——— A Simple Relation between Elementary Number Theory and Elementary Projective Geometry, 317.
LATHAM, MARCIA. The Astrolabe, 162.
LEHMER, D. N. See REVIEWS under Wilczynski.
LENNES, N. J. See under DISCUSSIONS.
MARTIN, E. N. See under DISCUSSIONS.
MILLER, G. A. Mathematics in the New International Encyclopaedia, 106.
——— The Obsolete in Mathematics, 453.
MITCHELL, U. G., Crathorne, A. R. and Garrett, W. H. Algebra Courses for College Juniors and Seniors, 362.
——— See under DISCUSSIONS.
MOORE, C. L. E. A Substitute for Dupin's Indicatrix, 456.
MORITZ, R. E. On the Construction of Certain Curves Given in Polar Coördinates, 213.
NYBERG, J. The Presentation of the Notion of Function, 309.
——— The Linear Function and the Line, 406.
PARKER, DEW. H. See REVIEWS under Richardson.
PETERSON, O. J. The Double Points of Rational Curves, 376.
PORTER, M. B. See under DISCUSSIONS.
PUTNAM, T. M. Mathematical Forms of certain Eroded Mountain Sides, 451.
RANSOM, W. R. See REVIEWS under Love.
REES, E. L. See under DISCUSSIONS.
REYNOLDS, J. B. See under DISCUSSIONS.
RIDER, P. R. See under DISCUSSIONS.
——— An Intrinsic Equation Solution of a Problem of Euler, 420.
ROBBINS, F. E. See REVIEWS under Johnson.
SCHMIEDEL, O. See under DISCUSSIONS.
SMITH, D. E. On the Origin of Certain Typical Problems, 64.
——— Mathematical Problems in Relation to the History of Economics and Commerce, 221.